QUANTIZATION OF THE HALL CONDUCTANCE AND DELOCALIZATION IN ERGODIC LANDAU HAMILTONIANS

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Abstract. We prove quantization of the Hall conductance for continuous ergodic Landau Hamiltonians under a condition on the decay of the Fermi projections. This condition and continuity of the integrated density of states are shown to imply continuity of the Hall conductance. In addition, we prove the existence of delocalization near each Landau level for these two-dimensional Hamiltonians. More precisely, we prove that for some ergodic Landau Hamiltonians there exists an energy E near each Landau level where a "localization length" diverges. For the Anderson-Landau Hamiltonian we also obtain a transition between dynamical localization and dynamical delocalization in the Landau bands, with a minimal rate of transport, even in cases when the spectral gaps are closed.

Contents

1. Introduction

Ergodic Landau Hamiltonians describe an electron moving in a very thin flat conductor with impurities under the influence of a constant magnetic field perpendicular to the plane of the conductor. They play an important role in the understanding of the quantum Hall effect [L, AoA, T, H, NT, Ku, Be, AvSS, BeES]. Laughlin's argument relies on the assumption that under weak disorder and strong magnetic field the energy spectrum consists of bands of extended states separated by energy regions of localized states and/or energy gaps [L, H, AoA, T]. Kunz [Ku]

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formulated assumptions under which he derived the divergence of a "localization length" near each Landau level at weak disorder.

Previous to our recent paper [GKS], there had been no rigorous results concerning delocalization for continuous ergodic Landau Hamiltonians. Divergence of a "localization length" had only been proved for an ergodic Landau Hamiltonian in a tight-binding approximation, a discrete ergodic Schrödinger operator. The first results were obtained by Bellissard, van Elst and Schulz-Baldes [BeES], who proved that, for an ergodic Landau Hamiltonian in a tight-binding approximation, if the Hall conductance jumps from one integer value to another between two Fermi energies, then there is an energy between these Fermi energies at which a certain localization length diverges. Their results relied on a proof of the quantization of Hall conductance (the quantum Hall effect) for ergodic Landau Hamiltonians in a tight binding representation (discrete ergodic Landau Hamiltonians) in energy intervals characterized by a condition on the decay of the Fermi projections. Their proof relies on noncommutative geometry and the Dixmier trace. Aizenman and Graf [AG] gave a more elementary derivation of this result, incorporating ideas of Avron, Seiler and Simon [AvSS], paying the price of a slightly stronger condition on the decay of the Fermi projections.

In [GKS] we proved that the (continuous) Anderson-Landau Hamiltonian (the random Landau Hamiltonian in [GKS]) exhibits dynamical delocalization in each Landau band. More precisely, under the disjoint bands condition (open spectral gaps between Landau bands), which holds (bounded potentials) under weak disorder and/or strong magnetic field, we proved the existence of a transition between dynamical localization and dynamical delocalization in each Landau band, with a lower bound on the rate of transport. We used nontrivial consequences of the multiscale analysis for random Schrödinger operators to prove that the Hall conductance for the Anderson-Landau Hamiltonian is well defined and constant in intervals of dynamical localization. We used the knowledge of the precise values of the Hall conductance for the (free) Landau Hamiltonian: it is constant between Landau levels and jumps by one at each Landau level, a well known fact (e.g., [AvSS, BeES]). In addition, we showed that the Hall conductance is constant as a function of the disorder parameter in the gaps between the Landau bands, a result previously derived by Elgart and Schlein [ES] for smooth potentials. Under the disjoint bands conditions (open spectral gaps), we combined these ingredients to conclude that there must be dynamical delocalization as we cross a Landau band. Moreover, since the existence of dynamical localization at the edges of these Landau bands was known [CoH, W, GK2], we proved the existence of dynamical mobility edges.

In [GKS] we circumvented the use of the quantization of the Hall conductance. For continuous Landau Hamiltonians quantization of the Hall conductance had only been known on spectral gaps [AvSS]. A proof of quantization of the Hall conductance inside the spectrum of continuous ergodic Landau Hamiltonians has been a long-standing open problem. Although it was promised in 1994 [BeES], the proof never appeared. (As mentioned in [BeES], in the discrete case their proof studies a compact noncommutative manifold, while in the continuous case the corresponding noncommutative manifold is locally compact, but not compact.)

In this article we prove quantization of the Hall conductance for continuous ergodic Landau Hamiltonians under a condition on the decay of the Fermi projections. We also show that this condition and continuity of the integrated density of states

imply continuity of the Hall conductance. In particular, we get quantization and continuity of the Hall conductance for the Anderson-Landau Hamiltonian in the region of localization.

Our condition on the decay of the Fermi projections is reminiscent of the condition used in [AG], but it is not the same because of differences between the continuous and the discrete cases. Although the weaker condition given in [BeES] is very natural (it was shown by Bouclet and the authors [BoGKS] to be sufficient for a rigorous derivation of the Kubo-Středa formula for the Hall conductance in continuous ergodic Landau Hamiltonians), its use for a derivation of the quantization of the Hall conductance seems to require methods of noncommutative geometry and the Dixmier trace that have not been extended to the continuous case.

In [GKS] we did not use the quantization of the Hall conductance, but required the disjoint bands condition. The results in this paper not only give a new proof of the delocalization results in [GKS], but they allow the extension of those results to ergodic Landau Hamiltonians, in the sense of divergence of a "localization length".

In this paper we go beyond the disjoint bands condition, proving dynamical delocalization in the Landau bands for the Anderson-Landau Hamiltonian in cases where the spectral gaps are closed. Using our results on the quantization of the Hall conductance, we prove the existence of a transition between dynamical localization and dynamical delocalization in a Landau band, with a lower bound on the rate of transport, for Anderson-Landau Hamiltonians with closed spectral gaps. Although in this paper we assume, as in [GKS], that the potentials are bounded, this restriction can be removed. This extension appears in a companion article [GKM], which considers an Anderson-Landau Hamiltonian with unbounded random amplitudes (e.g., with a Gaussian distribution), where all the gaps close as soon as the disorder is turned on. The main results of this paper still hold for such unbounded Anderson-Landau Hamiltonians; the theorem concerning the existence of a dynamical transition is stated below for completeness.

2. Definitions and main results

We consider a $\mathbb{Z}^2$-ergodic Landau Hamiltonian

$$H_{B,\lambda,\omega} = H_B + \lambda V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x), \tag{2.1}$$

where H_B is the (free) Landau Hamiltonian,

$$H_B = (-i\nabla - \mathbf{A})^2 \quad \text{with} \quad \mathbf{A} = \frac{B}{2}(x_2, -x_1) \tag{2.2}$$

($\mathbf{A}$ is the vector potential and $B > 0$ is the strength of the magnetic field, we use the symmetric gauge and incorporated the charge of the electron in the vector potential), $\lambda \geq 0$ is the disorder parameter, and V_ω is a bounded ergodic (real) potential. Thus, there is a probability space $(\Omega, \mathbb{P})$ equipped with an ergodic group $\{\tau(a);\ a \in \mathbb{Z}^2\}$ of measure preserving transformations, a potential-valued map V_ω on Ω, measurable in the sense that $\langle \phi, V_\omega \phi \rangle$ is a measurable function of ω for all $\phi \in C_c^\infty(\mathbb{R}^2)$. Such a family of potentials includes random as well as quasiperiodic potentials. We assume that

$$-M_1 \leq V_\omega(x) \leq M_2, \quad \text{where} \quad M_1, M_2 \in [0, \infty) \quad \text{with} \quad M_1 + M_2 > 0, \tag{2.3}$$

and

$$V_\omega(x - a) = V_{\tau_a \omega}(x) \quad \text{for all } a \in \mathbb{Z}^2. \tag{2.4}$$

An important example of an ergodic Landau Hamiltonian is the Anderson-Landau Hamiltonian

$$H^{(A)}_{B,\lambda,\omega} := H_B + \lambda V^{(A)}_\omega, \tag{2.5}$$

where $V^{(A)}_\omega$ is the random potential

$$V^{(A)}_\omega(x) = \sum_{i\in\mathbb{Z}^2} \omega_i\, u(x-i), \tag{2.6}$$

with $u(x) \geq 0$ a bounded measurable function with compact support, $u(x) \geq u_0$ on some nonempty open set for some constant $u_0 > 0$, and $\omega = \{\omega_i;\ i \in \mathbb{Z}^2\}$ a family of independent, identically distributed random variables taking values in a bounded interval $[-M_1, M_2]$ $(0 \leq M_1, M_2 < \infty,\ M_1 + M_2 > 0)$, whose common probability distribution μ has a bounded density ρ. Without loss of generality we set $\left\|\sum_{i\in\mathbb{Z}^2} u(x-i)\right\|_\infty = 1$, and hence $-M_1 \leq V^{(A)}_\omega(x) \leq M_2$.

An ergodic Landau Hamiltonian $H_{B,\lambda,\omega}$ is a self-adjoint measurable operator, i.e., with probability one $H_{B,\lambda,\omega}$ is a self-adjoint operator and the mappings $\omega \to f(H_{B,\lambda,\omega})$ are strongly measurable for all bounded measurable functions on $\mathbb{R}$ (cf. [PF]). The magnetic translations $U_a = U_a(B)$, $a \in \mathbb{R}^2$, defined by

$$\left(U_a\psi\right)(x) = \mathrm{e}^{-i\frac{B}{2}(x_2a_1 - x_1a_2)}\psi(x-a), \tag{2.7}$$

give a projective unitary representation of $\mathbb{R}^2$ on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x)$:

$$U_aU_b = \mathrm{e}^{i\frac{B}{2}(a_2b_1 - a_1b_2)}U_{a+b} = \mathrm{e}^{iB(a_2b_1-a_1b_2)}U_bU_a, \quad a, b \in \mathbb{Z}^2. \tag{2.8}$$

We have $U_aH_BU_a^* = H_B$ for all $a \in \mathbb{R}^2$, and the following covariance relation for magnetic translation by elements of $\mathbb{Z}^2$:

$$U_aH_{B,\lambda,\omega}U_a^* = H_{B,\lambda,\tau_a\omega} \quad \text{for all } a \in \mathbb{Z}^2. \tag{2.9}$$

It follows from ergodicity that that $H_{B,\lambda,\omega}$ has a nonrandom spectrum: there exists a nonrandom set $\Sigma_{B,\lambda}$ such that $\sigma(H_{B,\lambda,\omega}) = \Sigma_{B,\lambda}$ with probability one. Moreover the decomposition of $\sigma(H_{B,\lambda,\omega})$ into pure point spectrum, absolutely continuous spectrum, and singular continuous spectrum is also independent of the choice of ω with probability one [KiM1, CL, PF]. In addition, the integrated density of states $N(B, \lambda, E)$ is well defined and may be written as (cf. [HuLMW1])

$$N(B, \lambda, E) = \mathbb{E}\left\{\mathrm{tr}\left\{\chi_0 P_{B,\lambda,E,\omega}\chi_0\right\}\right\}. \tag{2.10}$$

Here and throughout the paper, χ_x denotes the characteristic function of a cube of side length 1 centered at $x \in \mathbb{Z}^2$.

The spectrum of the Landau Hamiltonian H_B, denoted by Σ_B, consists of a sequence of infinitely degenerate eigenvalues, the Landau levels:

$$\Sigma_B = \left\{B_n := (2n-1)B, \quad n = 1, 2, \ldots\right\}. \tag{2.11}$$

We also set $B_0 = -\infty$ for convenience. Standard arguments (see Appendix A) show that

$$\Sigma_{B,\lambda} \subset \bigcup_{n=1}^{\infty} \mathcal{B}_n(B,\lambda), \quad \text{where} \quad \mathcal{B}_n(B,\lambda) = [B_n - \lambda M_1, B_n + \lambda M_2]. \tag{2.12}$$

For a given magnetic field $B > 0$, disorder $\lambda \geq 0$ and energy $E \in \mathbb{R}$, the Fermi projection $P_{B,\lambda,E,\omega}$ is just the spectral projection of the ergodic Landau Hamiltonian $H_{B,\lambda,\omega}$ onto energies $\leq E$, i.e.,

$$P_{B,\lambda,E,\omega} = \chi_{(-\infty,E]}(H_{B,\lambda,\omega}). \tag{2.13}$$

Estimates on the decay of the operator kernel of the Fermi projection,

$$\{\chi_x P_{B,\lambda,E,\omega}\chi_y\}_{x,y\in\mathbb{Z}^2}\,,$$

play an important role in the study of the Hall conductance.

To state these estimates we introduce norms on random operators (see Subsection 3.1 for more details). A random operator S_ω is a strongly measurable map from the probability space $(\Omega,\mathbb{P})$ to bounded operators on $\mathrm{L}^2(\mathbb{R}^2,\mathrm{d}x)$. We set

$$\begin{aligned} &|||S_\omega|||_p := \{\mathbb{E}\{\operatorname{tr}|S_\omega|^p\}\}^{\frac{1}{p}} = \|\|S_\omega\|_p\|_{\mathrm{L}^p(\Omega,\mathbb{P})} \quad \text{for} \quad p\in[1,\infty),\\ &|||S_\omega|||_\infty := \|\|S_\omega\|\|_{\mathrm{L}^\infty(\Omega,\mathbb{P})}\,. \end{aligned} \tag{2.14}$$

The Hall conductance $\sigma_H(B,\lambda,E)$ is given by

$$\sigma_H(B,\lambda,E) = -2\pi i\,\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 P_{B,\lambda,E,\omega}\left[\left[P_{B,\lambda,E,\omega},X_1\right],\left[P_{B,\lambda,E,\omega},X_2\right]\right]\chi_0\right\}\right\}, \tag{2.15}$$

defined for $B>0$, $\lambda\ge 0$ and energy $E\in\mathbb{R}$ such that

$$|||\chi_0 P_{B,\lambda,E,\omega}\left[\left[P_{B,\lambda,E,\omega},X_1\right],\left[P_{B,\lambda,E,\omega},X_2\right]\right]\chi_0|||_1 < \infty. \tag{2.16}$$

(X_i denotes the operator given by multiplication by the coordinate x_i, $i=1,2$, and $|X|$ the operator given by multiplication by $|x|$.)

A natural condition for (2.16) and quantization of the Hall conductance was given by Bellissard et al [BeES]:

$$\sum_{x\in\mathbb{Z}^2}|x|^2\ |||\chi_x P_{B,\lambda,E,\omega}\chi_0|||_2^2<\infty. \tag{2.17}$$

They showed the sufficiency of this condition in an abstract C^*-algebra setting, from which they obtained existence and quantization of the Hall conductance for ergodic Landau Hamiltonians in a tight binding representation (ergodic Landau Hamiltonians). This condition was also shown by Bouclet and the authors [BoGKS] to be sufficient for a rigorous derivation of (2.15) for ergodic Landau Hamiltonians as a Kubo formula.

Aizenman and Graf [AG] gave a more elementary derivation of the existence and quantization of the Hall conductance for an ergodic Landau Hamiltonian $H_{B,\lambda,\omega}$ on $\ell^2(Z^2)$, under the condition [AG, condition (5.4)], namely

$$\sum_{x\in\mathbb{Z}^2}|x|\left\{\mathbb{E}\left\{|\langle\delta_x,P_{B,\lambda,E,\omega}\delta_0\rangle|^q\right\}\right\}^{\frac{1}{q}}<\infty \quad \text{for some} \quad q>2, \tag{2.18}$$

which implies (2.17) in the discrete setting.

In the discrete setting, given an interval where the integrated density of states is continuous, constancy of the Hall conductance follows if either (2.17) or (2.18) holds with a uniform bound in the interval [BeES, AG].

On the continuum, it is natural to work with estimates on the the decay of $|||\chi_x P_{B,\lambda,E,\omega}\chi_0|||_2$. In fact, it is known that for the Anderson-Landau Hamiltonian $|||\chi_x P_{B,\lambda,E,\omega}\chi_0|||_2$ exhibits sub-exponential in x in the region of localization [GK4, Theorem 3],[GKS, Eq. (3.2)]. We will prove that a sufficient condition for the existence and quantization of the Hall conductance for ergodic Landau Hamiltonians is given by

$$\sum_{x\in\mathbb{Z}^2}|x|\ |||\chi_x P_{B,\lambda,E,\omega}\chi_0|||_2^\beta<\infty \quad \text{for some} \quad \beta\in(0,1). \tag{2.19}$$

We will also show that for an interval where the integrated density of states is continuous, we have constancy of the Hall conductance if (2.19) holds with a locally bounded bound. Note that (2.19) implies (2.17).

We consider the magnetic field-disorder-energy parameter space

$$\Xi = \{(0,\infty) \times [0,\infty) \times \mathbb{R}\} \setminus \cup_{B\in(0,\infty)} \{(B,0)\times\Sigma_B\}; \tag{2.20}$$

we exclude the Landau levels at no disorder. We give Ξ the relative topology as a subset of $\mathbb{R}^3$. Given a subset $\Phi \subset \Xi$, we set

$$\Phi^{(B,\lambda)} := \{E \in \mathbb{R};\ (B,\lambda,E) \in \Phi\}, \tag{2.21}$$

with a similar definition for $\Phi^{(B,E)}$.

We now introduce a (generalized) "localization length" $L(B,\lambda,E)$, based on (2.19). Given $\beta \in (0,1]$ and $(B,\lambda,E)\in\Xi$, we set

$$L(B,\lambda,E) := \lim_{\beta\uparrow 1} L_\beta(B,\lambda,E), \tag{2.22}$$

where

$$L_\beta(B,\lambda,E) := \sum_{x\in\mathbb{Z}^2} |x|\ \left|\left|\left|\chi_x P_{B,\lambda,E,\omega}\chi_0\right|\right|\right|_2^\beta \quad \text{for} \quad \beta\in(0,1]. \tag{2.23}$$

We will also need 'localization lengths" that take into account what happens near (B,λ,E). We let

$$L_+(B,\lambda,E) := \lim_{\beta\uparrow 1} L_{\beta+}(B,\lambda,E), \tag{2.24}$$

$$L_+^{(B,\lambda)}(E) := \lim_{\beta\uparrow 1} L_{\beta+}^{(B,\lambda)}(E), \tag{2.25}$$

where

$$L_{\beta+}(B,\lambda,E) := \inf_{\substack{\Phi\ni(B,\lambda,E)\\ \Phi\subset\Xi \text{ open}}} \ \sup_{(B',\lambda',E')\in\Phi} L_\beta(B',\lambda',E'), \tag{2.26}$$

$$L_{\beta+}^{(B,\lambda)}(E) := \inf_{\substack{I\ni E\\ I\subset\mathbb{R} \text{ open}}} \ \sup_{E'\in I} L_\beta(B,\lambda,E'). \tag{2.27}$$

The justification of the definitions (2.22), (2.24) and (2.25), that is, the existence of the limits, is found in Subsection 3.3. Note that $L_1(B,\lambda,E)<\infty$ implies (2.17), and that in general we only have $L_1(B,\lambda,E)\le L(B,\lambda,E)$.

We also define the subsets of Ξ where these "localization lengths" are finite:

$$\begin{aligned} \Xi_\# &:= \{(B,\lambda,E)\in\Xi;\quad \#(B,\lambda,E)<\infty\}, \quad \# = L, L_+, L_\beta, L_{\beta+}, \\ \Xi_\#^{\{B,\lambda\}} &:= \left\{E\in\mathbb{R};\quad \#^{(B,\lambda)}(E)<\infty\right\}, \quad \# = L, L_+, L_\beta, L_{\beta+}. \end{aligned} \tag{2.28}$$

Ξ_{L_+} is, by definition, a relatively open subset of Ξ, and $\Xi_{L_+}^{\{B,\lambda\}}$ is an open subset of $\mathbb{R}$. Note that $\Xi_\#^{\{B,\lambda\}} \supset \Xi_\#^{(B,\lambda)}$, with $\Xi_\#^{(B,\lambda)}$ defined as in (2.21), but we may not have equality.

In Subsection 3.3 we show that the sets $\Xi_\#$ and $\Xi_\#^{\{B,\lambda\}}$, $\# = L_\beta, L_{\beta+}$, are monotone increasing in $\beta\in(0,1]$, with

$$\Xi_L = \bigcup_{\beta\in(0,1)} \Xi_{L_\beta}, \quad \Xi_{L_+} = \bigcup_{\beta\in(0,1)} \Xi_{L_{\beta+}}, \quad \Xi_{L_+}^{\{B,\lambda\}} = \bigcup_{\beta\in(0,1)} \Xi_{L_{\beta+}}^{\{B,\lambda\}}. \tag{2.29}$$

Note that

$$\Xi_{\mathrm{NS}} := \{(B,\lambda,E)\in\Xi;\ E\notin\Sigma_{B,\lambda}\} \subset \Xi_{L_+}; \tag{2.30}$$

$\Xi_{\rm NS}$ being the *region of no spectrum.*

We are now ready to state our main results.

Theorem 2.1. *Let $H_{B,\lambda,\omega}$ be an ergodic Landau Hamiltonian. Then the Hall conductance $\sigma_H(B,\lambda,E)$ is defined and integer valued on Ξ_L. In addition, $\sigma_H(B,\lambda,E)$ is locally bounded on Ξ_{L_+} and on each $\Xi_{L_+}^{\{B,\lambda\}}$.*

We set $\sigma_H^{(B,\lambda)}(E) := \sigma_H(B,\lambda,E)$, $N^{(B,\lambda)}(E) := N(B,\lambda,E)$, and $L_+^{(B,\lambda)}(E) := L_+(B,\lambda,E)$.

Theorem 2.2. *Let $H_{B,\lambda,\omega}$ be an ergodic Landau Hamiltonian. If for a given $(B,\lambda) \in (0,\infty)\times[0,\infty)$ the integrated density of states $N^{(B,\lambda)}(E)$ is continuous in E, then the Hall conductance $\sigma_H^{(B,\lambda)}(E)$ is continuous on $\Xi_{L_+}^{\{B,\lambda\}}$. In particular, $\sigma_H^{(B,\lambda)}(E)$ is constant on each connected component of $\Xi_{L_+}^{\{B,\lambda\}}$.*

If we have

$$\lambda(M_1+M_2) < 2B, \tag{2.31}$$

it follows from (2.12) that the bands $\mathcal{B}_n(B,\lambda)$ are disjoint, and the spectral gaps remain open. We will refer to (2.31) as the *disjoint bands condition*; it clearly holds under weak disorder and/or strong magnetic field.

Corollary 2.3. *Let $H_{B,\lambda,\omega}$ be an ergodic Landau Hamiltonian. Suppose the integrated density of states $N^{(B,\lambda)}(E)$ is continuous in E for all $(B,\lambda)\in(0,\infty)\times[0,\infty)$ satisfying the disjoint bands condition (2.31). Then for all such (B,λ) the "localization length" $L_+^{(B,\lambda)}(E)$ diverges near each Landau level: for each $n=1,2,\ldots$ there exists an energy $E_n(B,\lambda)\in\mathcal{B}_n(B,\lambda)$ such that*

$$L_+^{\{B,\lambda\}}(E_n(B,\lambda)) = \infty. \tag{2.32}$$

For the Anderson-Landau Hamiltonian $H_{B,\lambda,\omega}^{(A)}$ we can say more. Following [GK3, GK4, GKS] we introduce the region of dynamical localization. (It was called the strong insulator region in [GK3] and the region of complete localization in [GK4].) This can be done in many equivalent ways, as shown in [GK3, GK4], but for the purposes of this paper we define it by the decay of the Fermi projection, using [GK4, Theorem 3 and following comments]: The region of dynamical localization $\Xi_{\rm DL}$ consists of those $(B,\lambda,E)\in\Xi$ for which there exists an open interval $I\ni E$ such that

$$\sup_{E'\in I} |||\chi_x P_{B,\lambda,E',\omega}\chi_0|||_2 \le C_{I,B,\lambda}\,(1+|x|)^{-\eta_1} \quad \text{for all} \quad x\in\mathbb{Z}^2, \tag{2.33}$$

where $\eta_1>0$ is a fixed number that can be calculated from the proof of [GK4, Theorem 3]. (The condition stated in [GK4, Theorem 3] is of the form

$$\mathbb{E}\left\{\sup_{E'\in I} \|\chi_x P_{B,\lambda,E',\omega}\chi_0\|_2^2\right\} \le C_{I,B,\lambda}\,(1+|x|)^{-\eta_1} \quad \text{for all} \quad x\in\mathbb{Z}^2, \tag{2.34}$$

but an inspection of the proof shows that it can be replaced by (2.33).) Its complement in Ξ will be called the region of dynamical delocalization: $\Xi_{\rm DD} := \Xi\setminus\Xi_{\rm DL}$. (See [GKS] for background, definitions, and discussion.) It follows that that there exists $\beta_1\in(0,1)$ such that

$$\Xi_{\rm DL}^{(B,\lambda)} = \Xi_{L_{\beta_1 +}}^{\{B,\lambda\}} \subset \Xi_{L_+}^{\{B,\lambda\}}. \tag{2.35}$$

Moreover, the integrated density of states $N(B,\lambda,E)$ of the the Anderson-Landau Hamiltonian is jointly Hölder-continuous in (B,E) for $\lambda>0$ [CoHKR]. ($N(B,\lambda,E)$ is actually Lipshitz continuous in E for fixed (B,λ) [CoHK2].) Thus (2.32) implies [GKS, Eq. (2.20)], that is,

$$\Xi_{\mathrm{DD}}^{(B,\lambda)} \cap \mathcal{B}_n(B,\lambda) \neq \emptyset, \tag{2.36}$$

and hence Corollary 2.3 provides a new proof for [GKS, Theorems 2.1 and 2.2].

We actually have more. Using the characterization of Ξ_{DL} as the region of applicability of the multiscale analysis [GK3], we can get the constant $C_{I,B,\lambda}$ in (2.33) locally bounded in B and λ, obtaining

$$\Xi_{\mathrm{DL}} = \Xi_{L_{\beta_1 +}} \subset \Xi_{L_+}. \tag{2.37}$$

For the Anderson-Landau Hamiltonian we have a slightly stronger version of Theorems 2.1 and 2.2.

Theorem 2.4. *Let $H^{(A)}_{B,\lambda,\omega}$ be the Anderson-Landau Hamiltonian. Then the Hall conductance $\sigma_H(B,\lambda,E)$ is defined and integer valued on Ξ_L, and Hölder-continuous on Ξ_{L_+}. In particular, $\sigma_H(B,\lambda,E)$ is constant on each connected component of Ξ_{L_+}.*

It follows that on Ξ_{DL}, the region of dynamical localization , the Hall conductance $\sigma_H(B,\lambda,E)$ is defined, integer valued, and constant on each connected component .

The results in this article for the Anderson-Landau Hamiltonian go beyond [GKS, Theorems 2.1 and 2.2]; they show the existence of a dynamical metal-insulator transition, in the sense of [GK3], inside the Landau bands of the Anderson-Landau Hamiltonian in cases when the disjoint bands condition does not hold and the spectral gaps are closed. We give a simple example in the next theorem.

As shown in [GK3], the region of dynamical localization $\Xi_{\mathrm{DL}}^{(B,\lambda)}$ can be characterized as follows. To measure 'dynamical localization' we introduce

$$M_{B,\lambda,\omega}(p,\mathcal{X},t) = \left\| \langle x\rangle^{\frac{p}{2}} \mathrm{e}^{-itH_{B,\lambda,\omega}} \mathcal{X}(H_{B,\lambda,\omega})\chi_0 \right\|_2^2, \tag{2.38}$$

the random moment of order $p\geq 0$ at time t for the time evolution in the Hilbert-Schmidt norm, initially spatially localized in the square of side one around the origin (with characteristic function χ_0), and "localized" in energy by the function $\mathcal{X}\in C^\infty_{c,+}(\mathbb{R})$. (Notation: $\langle x\rangle := \sqrt{1+|x|^2}$.) Its time averaged expectation is given by

$$\mathcal{M}_{B,\lambda}(p,\mathcal{X},T) = \frac{1}{T}\int_0^\infty \mathbb{E}\left\{M_{B,\lambda,\omega}(p,\mathcal{X},t)\right\} \mathrm{e}^{-\frac{t}{T}}\,\mathrm{d}t. \tag{2.39}$$

It is proven in [GK3] that $\Xi_{\mathrm{DL}}^{(B,\lambda)}$ is the set of energies E for which there exists $\mathcal{X}\in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X}\equiv 1$ on some open interval containing E, $\alpha\geq 0$, and $p>4\alpha+22$, such that

$$\liminf_{T\to\infty} \frac{1}{T^\alpha}\mathcal{M}_{B,\lambda}(p,\mathcal{X},T) < \infty, \tag{2.40}$$

in which case it is also shown in [GK3] that (2.40) holds for any $p\geq 0$ with $\alpha=0$.

Theorem 2.5. *Let $H^{(A)}_{B,\lambda,\omega}$ be an Anderson-Landau Hamiltonian as in* (2.5)-(2.6), *where the common probability distribution μ has density*

$$\rho(s) = \tfrac{\eta+1}{2}\left(1-|s|\right)^\eta \chi_{[-1.1]}(s), \quad \eta>0, \tag{2.41}$$

and the single-site potential u satisfies

$$0 < U_- \le U(x) := \sum_{i\in\mathbb{Z}^2} u(x-i) \le 1, \quad \text{with } U_- \text{ a constant.} \tag{2.42}$$

Let $B > 0$. Then:
(i) The spectral gaps are all closed for $\lambda \ge \frac{1}{U_-}B$:

$$\Sigma_{B,\lambda} = [E_0(B,\lambda),\infty) \quad \text{for} \quad \lambda \ge \tfrac{1}{U_-}B, \tag{2.43}$$

where $E_0(B,\lambda) := \inf \Sigma_{B,\lambda} \in (B-\lambda, B-\lambda U_-)$.

(ii) Let $\widehat{\lambda} > \frac{1}{U_-}B$, and $\delta \in (0,B)$. Set

$$\begin{aligned} J_n(B) &:= (B_n+\delta, B_{n+1}-\delta), \quad n\in\mathbb{N}, \\ J_0(B) &:= (-\infty, B-\delta) \subset (-\infty, B). \end{aligned} \tag{2.44}$$

Then for all $N \in \mathbb{N}$ there exists $\eta_N > 0$ such that, taking $\eta \ge \eta_N$, for all $\lambda \in [0,\widehat{\lambda}]$ we have

$$J_n(B) \subset \Xi_{\mathrm{DL}}^{(B,\lambda)} \quad \text{for all} \quad \lambda \in [0,\widehat{\lambda}], \quad n = 0,1,2,\ldots,N. \tag{2.45}$$

Moreover, for all $\lambda \in [0,\widehat{\lambda}]$ there exists

$$E_n(B,\lambda) \in [B_n-\delta, B_n+\delta] \cap \Xi_{\mathrm{DD}}^{(B,\lambda)} \quad \text{for} \quad n = 1,2,\ldots,N. \tag{2.46}$$

In particular, for $n = 1,2,\ldots,N$ we have $L_+^{\{B,\lambda\}}(E_n(B,\lambda)) = \infty$, and for every $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \ni E_n(B,\lambda)$ and $p > 24$, we have

$$\mathcal{M}_{B,\lambda}(p,\mathcal{X},T) \ge \; C_{p,\mathcal{X}}\, T^{\frac{p}{4}-6} \tag{2.47}$$

for all $T \ge 0$ with $C_{p,\mathcal{X}} > 0$.

Note that for all $\lambda \in [\frac{1}{U_-}B, \widehat{\lambda}]$ all the spectral gaps are closed, but we still show existence of at least one dynamical mobility edge near the first N Landau levels, namely a boundary point between the regions of dynamical localization and dynamical delocalization.

Another application of the results in this paper can be found in a companion article [GKM], which considers an Anderson-Landau Hamiltonian $H^{(A)}_{B,\lambda,\omega}$ as in (2.5)-(2.6), but with a common probability distribution μ which has a bounded density ρ with $\operatorname{supp}\rho = \mathbb{R}$ and fast decay:

$$\rho(\omega) \le \rho_0 \exp(-|\omega|^\alpha) \quad \text{for some } \rho_0 \in (0,+\infty) \text{ and } \alpha > 0. \tag{2.48}$$

(In particular, μ may have a Gaussian distribution.) The random potential V_ω is now an unbounded ergodic potential, but $H_{B,\lambda,\omega}$ is essentially self-adjoint on $\mathcal{C}^\infty_c(\mathbb{R}^d)$ with probability one, and we have (see [BCH])

$$\Sigma_{B,\lambda} = \mathbb{R} \quad \text{for all} \quad \lambda > 0, \tag{2.49}$$

where $\Sigma_{B.\lambda}$ is the spectrum of $H_{B,\lambda,\omega}$ with probability one.

It is shown in [GKM] that the main results of this paper, and in particular Theorems 2.1, 2.2 and 2.4, as well as the relevant results from [GK3], hold for these Anderson-Landau Hamiltonians with $\operatorname{supp}\mu = \mathbb{R}$ (and hence unbounded potentials). Note that (2.37) is still valid, although its proof must be modified, taking into account that the Wegner estimate can be controled as $\lambda \to 0$ for intervals that do not contain Landau levels. The fact that the Landau gaps are immediately filled up as soon as the disorder is turned on implies that the approach used in [GKS]

and in Corollary 2.3 is not applicable. Proving the existence of a dynamical transition in that case requires the full set of conclusions of Theorem 2.4, namely that the Hall conductance is integer valued and continuous on connected components of Ξ_{L_+}, as used in the proof of Theorem 2.5. The continuity of the Hall conductance for arbitrary small λ (in order to let λ go to zero) given by Theorem 2.4 is required. A result similar to Theorem 2.5(ii) is proved in [GKM]: given $n \in \mathbb{N}$, there is at least one dynamical mobility edge near the first N Landau levels for small λ. It can be stated as follows.

Theorem 2.6 ([GKM]). *Let $H_{B,\lambda,\omega}$ be a random Landau Hamiltonian as in (2.5)-(2.6), but with a common probability distribution μ which has a bounded density ρ with $\operatorname{supp}\rho = \mathbb{R}$ and (2.48), so (2.49) holds for all $\lambda > 0$. Let $B > 0$. Then, for each $n \in \mathbb{N}$, there exists $\lambda(n) > 0$, such that for $\lambda \in (0, \lambda(n)]$ there exist $E_n^{(\pm)}(B,\lambda)$, with $B_n - B < E_n^{(-)}(B,\lambda) < B_n < E_n^{(+)}(B,\lambda) < B_n + B$,*

$$\left| E_n^{(\pm)}(B,\lambda) - B_n \right| \le K_n(B)\lambda \left|\log\lambda\right|^{\frac{1}{\alpha}} \to 0 \quad \text{as } \lambda \to 0, \tag{2.50}$$

with a finite constant $K_n(B)$, and

$$\left(E_n^{(+)}(B,\lambda), (E_{n+1}^{(-)}(B,\lambda) \right) \subset \Xi_{\mathrm{DL}}^{(B,\lambda)}. \tag{2.51}$$

We also have $\left(-\infty, E_1^{(-)}(B,\lambda)\right) \subset \Xi_{\mathrm{DL}}^{(B,\lambda)}$ for $\lambda \in (0, \lambda(0)]$, $\lambda(0) > 0$.

Moreover, for $\lambda \in (0, \min\{\lambda(n-1), \lambda(n)\})$ there exists

$$E_n(B,\lambda) \in \left[E_n^{(-)}(B,\lambda), E_n^{(+)}(B,\lambda) \right] \cap \Xi_{\mathrm{DD}}^{(B,\lambda)}, \tag{2.52}$$

and hence (2.47) holds for every $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \ni E_n(B,\lambda)$ and $p > 24$.

We collect some technicalities in Section 3. In Section 4 we study the Hall conductance, proving Theorem 2.1. Section 5 is devoted to the continuity of the Hall conductance: Theorem 2.2 is proved in Subsection 5.1, and the stronger version for Anderson-Landau Hamiltonians, Theorem 2.4, is proved in Subsection 5.2. Corollary 2.3 is proven in Section 6. Dynamical delocalization (and a dynamical metal-insulator transition) for Anderson-Landau Hamiltonians with closed spectral gaps is shown in Section 7, where we prove Theorem 2.5. In Appendix A we prove a useful lemma about the spectrum of Landau Hamiltonians with bounded potentials. The spectrum of the Anderson-Landau Hamiltonian is discussed in Appendix B.

3. Technicalities

3.1. **Norms on random operators and Fermi projections.** Given $p \in [1,\infty)$, $\mathcal{T}_p$ will denote the Banach space of bounded operators S on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x)$ with $\|S\|_{\mathcal{T}_p} = \|S\|_p := (\operatorname{tr}|S|^p)^{\frac{1}{p}} < \infty$. A random operator S_ω is a strongly measurable map from the probability space $(\Omega, \mathbb{P})$ to bounded operators on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x)$. Given $p \in [1,\infty)$, we set

$$|||S_\omega|||_p := \left\{ \mathbb{E}\left\{ \|S_\omega\|_p^p \right\} \right\}^{\frac{1}{p}} = \left\| \|S_\omega\|_{\mathcal{T}_p} \right\|_{\mathrm{L}^p(\Omega,\mathbb{P})}, \tag{3.1}$$

and

$$|||S_\omega|||_\infty := \left\| \|S_\omega\| \right\|_{\mathrm{L}^\infty(\Omega,\mathbb{P})}. \tag{3.2}$$

These are norms on random operators, note that

$$|||S_\omega|||_q \le |||S_\omega|||_\infty^{\frac{q-p}{q}}\, |||S_\omega|||_p^{\frac{p}{q}} \quad \text{for } 1 \le p \le q < \infty, \tag{3.3}$$

and they satisfy Holder's inequality:

$$|||S_\omega T_\omega|||_r \le |||S_\omega|||_p\, |||T_\omega|||_q \quad \text{for } r,p,q \in [1,\infty] \text{ with } \tfrac{1}{r} = \tfrac{1}{p} + \tfrac{1}{q}. \tag{3.4}$$

In particular, if $|||S_\omega|||_\infty \le 1$, we have

$$|||S_\omega|||_q \le |||S_\omega|||_2^{\frac{2}{q}} \quad \text{for } 2 \le p \le q < \infty, \tag{3.5}$$

3.2. **Operator kernels of Fermi projections.** Let $H_{B,\lambda,\omega}$ be an ergodic Landau Hamiltonian for a given magnetic field $B > 0$, disorder $\lambda \ge 0$ and energy $E \in \mathbb{R}$. We consider the operator kernel of the Fermi projection $P_{B,\lambda,E,\omega} = \chi_{(-\infty,E]}(H_{B,\lambda,\omega})$, $\{\chi_x P_{B,\lambda,E,\omega}\chi_y\}_{x,y\in\mathbb{Z}^2}$, and set

$$\begin{aligned} \kappa_p(B,\lambda,E) &\equiv |||\chi_0 P_{B,\lambda,E,\omega}\chi_0|||_p \quad \text{for } p \in [1,\infty], \\ \kappa_{1,\infty}(B,\lambda,E) &\equiv \|\mathrm{tr}\,\{\chi_0 P_{B,\lambda,E,\omega}\chi_0\}\|_{\mathrm{L}^\infty(\Omega,\mathbb{P})}\,. \end{aligned} \tag{3.6}$$

Note that $\kappa_{1,\infty}(B,\lambda,E)$ is locally bounded on Ξ (e.g., [BoGKS]), and hence also $\kappa_p(B,\lambda,E)$, since $\kappa_\infty(B,\lambda,E) \le 1$ and for $p \in [1,\infty)$ we have

$$\kappa_p(B,\lambda,E) \le |||\chi_0 P_{B,\lambda,E,\omega}\chi_0|||_1^{\frac{1}{p}} \le \{\kappa_{1,\infty}(B,\lambda,E)\}^{\frac{1}{p}}\,. \tag{3.7}$$

In addition, we have

$$|||\chi_0 P_{B,\lambda,E,\omega}|||_p \begin{cases} = |||\chi_0 P_{B,\lambda,E,\omega}\chi_0|||_{\frac{p}{2}}^{\frac{1}{2}} = \left\{\kappa_{\frac{p}{2}}(B,\lambda,E)\right\}^{\frac{1}{2}} & \text{if } p \in [2,\infty) \\ = \left|\left|\left|\,|\chi_0 P_{B,\lambda,E,\omega}|^{\frac{1}{2}}\right|\right|\right|_{2p}^{2} \le \kappa_p(B,\lambda,E) & \text{if } p \in [1,\infty) \end{cases}, \tag{3.8}$$

and thus, given $x \in \mathbb{Z}^2$, for all $p \in [1,\infty)$ we have

$$|||\chi_0 P_{B,\lambda,E,\omega}\chi_x|||_p \le |||\chi_0 P_{B,\lambda,E,\omega}|||_{2p}\, |||P_{B,\lambda,E,\omega}\chi_x|||_{2p} = \kappa_p(B,\lambda,E). \tag{3.9}$$

It follows from (2.10) that

$$N(B,\lambda,E) = \kappa_1(B,\lambda,E). \tag{3.10}$$

Note that

$$N(B,\lambda,E) = 0 \quad \Longleftrightarrow \quad |||\chi_x P_{B,\lambda,E,\omega}\chi_0|||_2 = 0 \quad \text{for all} \quad x \in \mathbb{Z}^2. \tag{3.11}$$

3.3. **"Localization lengths".** We will now justify the definitions (2.22), (2.24) and (2.25).

To justify (2.22), we must show that the limit exists in $[0,\infty)$. Given $\beta \in (0,1]$ and $(B,\lambda,E) \in \Xi$, let

$$\widetilde{L}_\beta(B,\lambda,E) := N(B,\lambda,E)^{1-\beta} L_\beta(B,\lambda,E), \tag{3.12}$$

where $N(B,\lambda,E)$ is as in (2.10). It follows from (3.9) that $\widetilde{L}_\beta(B,\lambda,E)$ is monotone decreasing in $\beta \in (0,1]$, so we can define

$$\widetilde{L}(B,\lambda,E) := \inf_{\beta\in(0,1)} \widetilde{L}_\beta(B,\lambda,E) = \lim_{\beta\uparrow 1} \widetilde{L}_\beta(B,\lambda,E). \tag{3.13}$$

It is an immediate consequence of (3.12) and (3.13) (cf. (3.11)) that $L(B,\lambda,E)$ is well defined and

$$L(B,\lambda,E) = \widetilde{L}(B,\lambda,E). \tag{3.14}$$

The definitions (2.24) and (2.25) are justified in a similar way. As before

$$\begin{aligned}\widetilde{L}_{\beta+}(B,\lambda,E) &:= N(B,\lambda,E)^{1-\beta}L_{\beta+}(B,\lambda,E),\\ \widetilde{L}_{\beta+}^{(B,\lambda)}(E) &:= N(B,\lambda,E)^{1-\beta}L_{\beta+}^{(B,\lambda)}(E),\end{aligned}\tag{3.15}$$

are seen to be monotone decreasing in $\beta \in (0,1]$, so we have

$$L_{+}(B,\lambda,E) = \inf_{\beta\in(0,1)} \widetilde{L}_{\beta+}(B,\lambda,E) = \lim_{\beta\uparrow 1} \widetilde{L}_{\beta+}(B,\lambda,E),\tag{3.16}$$

$$L_{+}^{(B,\lambda)}(E) = \inf_{\beta\in(0,1)} \widetilde{L}_{\beta+}^{(B,\lambda)}(E) = \lim_{\beta\uparrow 1} \widetilde{L}_{\beta+}^{(B,\lambda)}(E).\tag{3.17}$$

It follows that that the sets $\Xi_{\#}$ and $\Xi_{\#}^{\{B,\lambda\}}$, $\# = L_\beta, L_{\beta+}$, are monotone increasing in $\beta \in (0,1]$, and we have (2.29)

3.4. **Auxiliary "localization lengths".** Although the "localization lengths" $L(B,\lambda,E)$ and $L_{+}(B,\lambda,E)$ give a convenient way to write our main theorems, in the proofs it will be more convenient to work with auxiliary "localization lengths" based on the norms for random operators introduced in (2.14) with $p \in [2,\infty)$. They can be thought of an adaptation to the continuum (and to two parameters) of [AG, condition (5.4)]. If $q \in [1,\infty)$, $J \subset [1,\infty)$, we define the following "localization lengths" for $(B,\lambda,E) \in \Xi$:

$$\begin{aligned}\ell_q(B,\lambda,E) &:= \sum_{x\in\mathbb{Z}^2} \max\{|x|,1\} \left|\!\left|\!\left| \chi_x P_{B,\lambda,E,\omega}\chi_0 \right|\!\right|\!\right|_q,\\ \ell_{q+}(B,\lambda,E) &:= \inf_{\substack{\Phi\ni(B,\lambda,E)\\ \Phi\subset\Xi \text{ open}}} \ \sup_{(B',\lambda',E')\in\Phi} \ell_q(B',\lambda',E'),\\ \ell_{q+}^{(B,\lambda)}(E) &:= \inf_{\substack{I\ni E\\ I\subset\mathbb{R}\text{ open}}} \ \sup_{E'\in I} \ell_q(B,\lambda,E'),\\ \ell_J(B,\lambda,E) &:= \inf_{q\in J} \ell_q(B,\lambda,E),\\ \ell_{J+}(B,\lambda,E) &:= \inf_{q\in J} \ell_{q+}(B,\lambda,E),\\ \ell_{J+}^{(B,\lambda)}(E) &:= \inf_{q\in J} \ell_{q+}^{(B,\lambda)}(E).\end{aligned}\tag{3.18}$$

While the quantity in [AG, (5.4)] is monotone increasing in $q \in [1,\infty)$, the "localization lengths" $\ell_q(B,\lambda,E)$ cannot be compared for different q's. Another difference is that [AG, condition (5.4)] implies the equivalent of (2.17) in the lattice, but $\ell_q(B,\lambda,E) < \infty$ only implies (2.17) if $q = 2$.

We also define the subsets of Ξ where these "localization lengths" are finite:

$$\begin{aligned}\Xi_{\#} &= \{(B,\lambda,E)\in\Xi;\quad \ell_{\#}(B,\lambda,E)<\infty\}, \quad \# = q, q+, J, J+,\\ \Xi_{\#}^{\{B,\lambda\}} &= \left\{E\in\mathbb{R};\quad \ell_{\#}^{(B,\lambda)}(E)<\infty\right\}, \quad \# = q+, J+\ .\end{aligned}\tag{3.19}$$

Note that we may have $\Xi_{\#}^{\{B,\lambda\}} \neq \Xi_{\#}^{(B,\lambda)}$, with $\Xi_{\#}^{(B,\lambda)}$ defined as in (2.21). However, $\Xi_{\#}^{\{B,\lambda\}} \supset \Xi_{\#}^{(B,\lambda)}$ and

$$\Xi_J = \bigcup_{q\in J} \Xi_q\,, \qquad \Xi_{J+} = \bigcup_{q\in J} \Xi_{q+}\,, \qquad \Xi_{J+}^{\{B,\lambda\}} = \bigcup_{q\in J} \Xi_{q+}^{\{B,\lambda\}}\,.\tag{3.20}$$

Ξ_{J+} is, by definition, a relatively open subset of Ξ, and $\Xi_{J+}^{\{B,\lambda\}}$ is an open subset of $\mathbb{R}$.

If $q \in [2,\infty)$, it follows immediately from (3.5) and (3.6) that for all $(B,\lambda,E) \in \Xi$ we have

$$\ell_q(B,\lambda,E) \le \kappa_q(B,\lambda,E) + L_{\frac{2}{q}}(B,\lambda,E), \tag{3.21}$$

$$\ell_{q+}(B,\lambda,E) \le \kappa_q(B,\lambda,E) + L_{\frac{2}{q}+}(B,\lambda,E), \tag{3.22}$$

$$\ell_{q+}^{(B,\lambda)}(E) \le \kappa_q(B,\lambda,E) + L_{\frac{2}{q}+}^{(B,\lambda)}(E). \tag{3.23}$$

It follows that

$$\Xi_L \subset \bigcap_{r>2} \Xi_{(2,r]} \quad \text{and} \quad \Xi_{L_+} \subset \bigcap_{r>2} \Xi_{(2,r]+}. \tag{3.24}$$

For the Anderson-Landau Hamiltonian $H_{B,\lambda,\omega}^{(A)}$ the following holds for all large q_0 (recall (2.33)-(2.37)):

$$\begin{aligned} \Xi_{\mathrm{DL}} &= \bigcap_{q\in[1,\infty)} \Xi_{q+} = \Xi_{q_0+}, \\ \Xi_{\mathrm{DL}}^{(B,\lambda)} &= \bigcap_{q\in[1,\infty)} \Xi_{q+}^{\{B,\lambda\}} = \Xi_{q_0+}^{\{B,\lambda\}}. \end{aligned} \tag{3.25}$$

4. Existence and quantization of the Hall conductance

Theorem 2.1 is an immediate consequence of the following theorem.

Theorem 4.1. *Let $H_{B,\lambda,\omega}$ be an ergodic Landau Hamiltonian. Then the Hall conductance $\sigma_H(B,\lambda,E)$ is defined on $\Xi_{[2,\infty)}$ with the bound*

$$|\sigma_H(B,\lambda,E)| \le 4\pi \inf_{\substack{q\in[2,\infty)\\ \frac{1}{p}+\frac{2}{q}=1}} \left\{ \kappa_p(B,\lambda,E) \left\{\ell_q(B,\lambda,E)\right\}^2 \right\} < \infty. \tag{4.1}$$

It follows that $\sigma_H(B,\lambda,E)$ is locally bounded on $\Xi_{[2,\infty)+}$ and on each $\Xi_{[2,\infty)+}^{\{B,\lambda\}}$. Moreover, the Hall conductance $\sigma_H(B,\lambda,E)$ is integer valued on $\Xi_{(2,3]}$.

Theorem 4.1 will proved by the following lemmas.

Given $x \in \mathbb{R}^2$, we set $\hat{x}$ to be the discretization of x, i.e., the unique element of $\mathbb{Z}^2$ such that $x_i \in [\hat{x}_i - \frac{1}{2}, \hat{x}_i + \frac{1}{2})$, $1 = 1,2$. We let $\hat{X}_i$ denote the operator given by multiplication by $\hat{x}_i$, and note that $\hat{X}_i\chi_u = u_i\chi_u$ for each $u \in \mathbb{Z}^2$, i.e., $\hat{X}_i = \sum_{x\in\mathbb{Z}^2} x\chi_x$, and note

$$\left\| X_i - \hat{X}_i \right\| \le \frac{1}{2}, \quad \left\| |X| - |\hat{X}| \right\| \le \frac{\sqrt{2}}{2}. \tag{4.2}$$

If $(B,\lambda,E) \in \Xi$ and $q \in [1,\infty)$, it follows that

$$\left|\!\left|\!\left| |\hat{X}|\, P_{B,\lambda,E,\omega}\chi_0 \right|\!\right|\!\right|_q \le \ell_q(B,\lambda,E), \tag{4.3}$$

and hence, using (4.2), and (3.8) we get

$$\left|\!\left|\!\left| |X|\, P_{B,\lambda,E,\omega}\chi_0 \right|\!\right|\!\right|_q \le \ell_q(B,\lambda,E) + \kappa_q(B,\lambda,E) \le 2\ell_q(B,\lambda,E). \tag{4.4}$$

It follows that, with $i = 1,2$,

$$\left|\!\left|\!\left| [P_{B,\lambda,E,\omega}, \hat{X}_i]\chi_0 \right|\!\right|\!\right|_q \le \ell_q(B,\lambda,E), \tag{4.5}$$

$$\left|\!\left|\!\left| [P_{B,\lambda,E,\omega}, X_i]\chi_0 \right|\!\right|\!\right|_q \le 3\ell_q(B,\lambda,E). \tag{4.6}$$

We conclude, using covariance, that for $\mathbb{P}$-a.e. ω, $\hat{X}_i P_{B,\lambda,E,\omega}\chi_u$ and $X_i P_{B,\lambda,E,\omega}\chi_u$, and hence also $[P_{B,\lambda,E,\omega},\hat{X}_i]\chi_u$ and $[P_{B,\lambda,E,\omega},X_i]\chi_u$, are bounded operators for all $(B,\lambda,E)\in\Xi_{[1,\infty)}$, $u\in\mathbb{Z}^2$, $i=1,2$.

We now define a modified Hall conductance, with $\hat{X}_i$ substituted for X_i:

$$\hat{\sigma}_H(\lambda,E) = -2\pi i\,\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 P_{B,\lambda,E,\omega}\left[\left[P_{B,\lambda,E,\omega},\hat{X}_1\right],\left[P_{B,\lambda,E,\omega},\hat{X}_2\right]\right]\chi_0\right\}\right\}, \tag{4.7}$$

defined for $(B,\lambda,E)\in\Xi$ such that

$$\left|\!\left|\!\left|\chi_0 P_{B,\lambda,E,\omega}\left[\left[P_{B,\lambda,E,\omega},\hat{X}_1\right],\left[P_{B,\lambda,E,\omega},\hat{X}_2\right]\right]\chi_0\right|\!\right|\!\right|_1 < \infty. \tag{4.8}$$

Lemma 4.2. *The Hall conductances $\sigma_H(B,\lambda,E)$ and $\hat{\sigma}_H(B,\lambda,E)$ are defined on the set $\Xi_{[2,\infty)}$. Moreover, for all $(B,\lambda,E)\in\Xi_{[2,\infty)}$ we have*

$$\begin{aligned}\sigma_H(B,\lambda,E) &= \hat{\sigma}_H(B,\lambda,E) \\ &= -2\pi i\sum_{u,v\in\mathbb{Z}^2}(u_1v_2-u_2v_1)\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 P_{B,\lambda,E,\omega}\chi_u P_{B,\lambda,E,\omega}\chi_v P_{B,\lambda,E,\omega}\chi_0\right\}\right\},\end{aligned} \tag{4.9}$$

with

$$\begin{aligned}|\sigma_H(B,\lambda,E)| &\le 4\pi\sum_{u,v\in\mathbb{Z}^2}|u|\,|v|\,|\!|\!|\chi_0 P_{\lambda,E,\omega}\chi_u P_{\lambda,E,\omega}\chi_v P_{\lambda,E,\omega}\chi_0|\!|\!|_1 \\ &\le 4\pi\kappa_p(B,\lambda,E)\left\{\ell_q(B,\lambda,E)\right\}^2<\infty\end{aligned} \tag{4.10}$$

for all $q\in[2,\infty)$ and $\frac{1}{p}+\frac{2}{q}=1$.

Proof. Let $(B,\lambda,E)\in\Xi_q$ for some $q\in[1,\infty)$. Writing P_ω for $P_{B,\lambda,E,\omega}$, we have

$$\begin{aligned}&|\!|\!|\chi_0 P_\omega\left[[P_\omega,X_1],[P_\omega,X_2]\right]\chi_0|\!|\!|_1 \le \\ &\sum_{u\in\mathbb{Z}^2}\left\{|\!|\!|\chi_0 P_\omega[P_\omega,X_1]\chi_u[P_\omega,X_2]\chi_0|\!|\!|_1 + |\!|\!|\chi_0 P_\omega[P_\omega,X_2]\chi_u[P_\omega,X_1]\chi_0|\!|\!|_1\right\}<\infty,\end{aligned} \tag{4.11}$$

since may use the Holder's inequality (3.4) with $\frac{1}{p}+\frac{2}{q}=1$ to get

$$\begin{aligned}&\sum_{u\in\mathbb{Z}^2}|\!|\!|\chi_0 P_\omega[P_\omega,X_i]\chi_u[P_\omega,X_j]\chi_0|\!|\!|_1 \\ &\quad\le |\!|\!|\chi_0 P_\omega|\!|\!|_p\sum_{u\in\mathbb{Z}^2}|\!|\!|[P_\omega,X_i]\chi_u|\!|\!|_q(|u|+1)\,|\!|\!|\chi_u P_\omega\chi_0|\!|\!|_q \\ &\quad\le |\!|\!|\chi_0 P_\omega|\!|\!|_p\,|\!|\!|[P_\omega,X_i]\chi_0|\!|\!|_q\sum_{u\in\mathbb{Z}^2}(|u|+1)\,|\!|\!|\chi_u P_\omega\chi_0|\!|\!|_q \\ &\quad\le 4\kappa_p(B,\lambda,E)\left\{\ell_q(B,\lambda,E)\right\}^2<\infty\end{aligned} \tag{4.12}$$

for $i.j=1,2$, where we used covariance, (3.8), (4.6), and (3.18). Thus $\sigma_H(B,\lambda,E)$ is defined on the set Ξ_q, and similarly for $\hat{\sigma}_H(B,\lambda,E)$.

We will now show that $\sigma_H(B,\lambda,E)=\hat{\sigma}_H(B,\lambda,E)$. To see that, note that

$$\begin{aligned}&\sigma_H(B,\lambda,E)-\hat{\sigma}_H(B,\lambda,E) = \\ &\quad -2\pi i\,\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 P_\omega\left[\left[P_\omega,X_1-\hat{X}_1\right],[P_\omega,X_2]\right]\chi_0\right\}\right\} \\ &\qquad +2\pi i\,\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 P_\omega\left[\left[P_\omega,\hat{X}_1\right],\left[P_\omega,X_2-\hat{X}_2\right]\right]\chi_0\right\}\right\}.\end{aligned} \tag{4.13}$$

We have

$$\begin{aligned}
&\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 P_\omega\left[\left[P_\omega, X_1-\hat{X}_1\right],[P_\omega, X_2]\right]\chi_0\right\}\right\}\\
&\quad=\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 P_\omega(X_1-\hat{X}_1)\,(1-P_\omega)\,[P_\omega,X_2]\,\chi_0\right\}\right\} &(4.14)\\
&\qquad+\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0\,[P_\omega,X_2]\,(1-P_\omega)\,(X_1-\hat{X}_1)P_\omega\chi_0\right\}\right\}\\
&\quad=\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0(X_1-\hat{X}_1)\,(1-P_\omega)\,[P_\omega,X_2]\,P_\omega\chi_0\right\}\right\} &(4.15)\\
&\qquad+\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0(X_1-\hat{X}_1)P_\omega\,[P_\omega,X_2]\,(1-P_\omega)\,\chi_0\right\}\right\}\\
&\quad=\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0(X_1-\hat{X}_1)\,[P_\omega,X_2]\,\chi_0\right\}\right\}=0, &(4.16)
\end{aligned}$$

where in (4.16) we used centrality of trace, justified since $X_2\chi_0$ is a bounded operator, to go from (4.15) to (4.16) we used

$$(1-P_\omega)[P_\omega,X_2]P_\omega+P_\omega[P_\omega,X_2](1-P_\omega)=[P_\omega,X_2], \tag{4.17}$$

and the passage from (4.14) to (4.15) can be justified as follows:

$$\begin{aligned}
&\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 P_\omega(X_1-\hat{X}_1)\,(1-P_\omega)\,[P_\omega,X_2]\,\chi_0\right\}\right\}\\
&\quad=\sum_{u\in\mathbb{Z}^2}\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 P_\omega\chi_u(X_1-\hat{X}_1)\,(1-P_\omega)\,[P_\omega,X_2]\,\chi_0\right\}\right\}\\
&\quad=\sum_{u\in\mathbb{Z}^2}\mathbb{E}\left\{\operatorname{tr}\left\{\chi_u(X_1-\hat{X}_1)\,(1-P_\omega)\,[P_\omega,X_2]\,\chi_0 P_\omega\chi_u\right\}\right\} &(4.18)\\
&\quad=\sum_{u\in\mathbb{Z}^2}\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0(X_1-\hat{X}_1)\,(1-P_\omega)\,[P_\omega,X_2]\,\chi_{-u}P_\omega\chi_0\right\}\right\}\\
&\quad=\mathbb{E}\left\{\operatorname{tr}\left\{\chi_0(X_1-\hat{X}_1)\,(1-P_\omega)\,[P_\omega,X_2]\,P_\omega\chi_0\right\}\right\},
\end{aligned}$$

with a similar calculation for the other term in (4.15), where we used the centrality of the trace and covariance (the absolute summability of all series can be verified as in (4.12)). The second term in the right-hand-side of (4.13) is also equal to 0 by a similar calculation, so we conclude that $\sigma_H(B,\lambda,E)=\hat{\sigma}_H(B,\lambda,E)$.

Since, with $\frac{1}{p}+\frac{2}{q}=1$, we have

$$|u||v|\,|||\chi_0 P_\omega\chi_u P_\omega\chi_v P_\omega\chi_0|||_1 \le |u|\,|||\chi_0 P_\omega\chi_u|||_q\,|||\chi_0 P_\omega|||_p\,|v|\,|||\chi_v P_\omega\chi_0|||_q\,, \tag{4.19}$$

the estimate (4.10) follows from (3.18) and (3.8). The expression (4.9) then follows for $\sigma_H(B,\lambda,E)=\hat{\sigma}_H(B,\lambda,E)$ from (4.7). □

Next, we will show that the Hall conductance $\sigma_H(\lambda,E)$ takes integer values on $\Xi_{(2,3]}$, following the approach of Avron, Seiler and Simon [AvSS], as modified by Aizenman and Graf [AG]. Avron, Seiler and Simon proved the result for random Landau Hamiltonians at energies outside the spectrum, i.e., on Ξ_{NS}. Their argument was adapted to the lattice by Aizenman and Graf, who proved that the Hall conductance for the lattice model takes integer values in the region where [AG, condition (5.4)] holds, i.e., on the lattice equivalent of $\Xi_{(2,3]}$. (On the lattice this result had been proved earlier under the lattice equivalent of condition (2.17) by Bellissard, van Elst and Schulz-Baldes [BeES].) We complete the circle by adapting Aizenman and Graf's argument back to the continuum.

Let $\mathbb{Z}^{2*} = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ denote the dual lattice to $\mathbb{Z}^2$. Given $a \in \mathbb{Z}^{2*}$ we define the complex valued function $\gamma_a(x)$ on $\mathbb{R}^2$ by

$$\gamma_a(x) = \frac{\hat{x}_1 - a_1 + i(\hat{x}_2 - a_2)}{|\hat{x} - a|}, \tag{4.20}$$

and let Γ_a denote the unitary operator given by multiplication by the function $\gamma_a(x)$. Note that $|\hat{x} - a| \geq \frac{\sqrt{2}}{2}$ for all $x \in \mathbb{R}^2$. We have the following estimate:

$$|\gamma_a(x) - \gamma_a(y)| \leq \min\left\{ |\hat{x} - \hat{y}| \max\left\{ \frac{1}{|\hat{x} - a|}, \frac{1}{|\hat{y} - a|} \right\}, 2 \right\} \leq \min\left\{ 4\frac{|\hat{x} - \hat{y}|}{|\hat{x} - a|}, 2 \right\}. \tag{4.21}$$

(The first inequality can be found in [AvSS]. The second inequality can be seen as follows: if $|\hat{x} - \hat{y}| \leq \frac{1}{2}|\hat{x} - a|$ we have $|\hat{x} - a| - |\hat{y} - a| \leq |\hat{x} - \hat{y}| \leq \frac{1}{2}|\hat{x} - a|$, and hence $|\hat{x} - a| \leq 2|\hat{y} - a|$; if $|\hat{x} - \hat{y}| > \frac{1}{2}|\hat{x} - a|$ we have $\frac{|\hat{x} - \hat{y}|}{|\hat{x} - a|} > \frac{1}{2}$, and hence $4\frac{|\hat{x} - \hat{y}|}{|\hat{x} - a|} > 2$.)

Given two orthogonal projections P and Q in a Hilbert space, such that $P - Q$ is compact, the index of P and Q is defined by (cf. [AvSS, Section 2])

$$\operatorname{Index}(P, Q) := \dim \operatorname{Ker}(P - Q - 1) - \dim \operatorname{Ker}(Q - P - 1). \tag{4.22}$$

The index is a well defined integer since $P - Q$ compact implies that $\dim \operatorname{Ker}(P - Q \pm 1)$ are both finite. Note that in the case P and Q have finite rank we have

$$\operatorname{Index}(P, Q) = \dim \operatorname{Ran} P - \dim \operatorname{Ran} Q = \operatorname{tr}(P - Q). \tag{4.23}$$

Lemma 4.3. *The Hall conductance $\sigma_H(B, \lambda, E)$ takes integer values on $\Xi_{(2,3]}$.*

Proof. Let $(B, \lambda, E) \in \Xi_q$ for some $q \in (2, 3]$, and write P_ω for $P_{B,\lambda,E,\omega}$. As in [AvSS, AG], we prove that for all $a \in \mathbb{Z}^{2*}$ we have

$$\mathbb{E}\left(\|P_\omega - \Gamma_a P_\omega \Gamma_a^*\|_3\right) < \infty, \tag{4.24}$$

and hence for $\mathbb{P}$-a.e. ω the index of the orthogonal projections P_ω and $\Gamma_a P_\omega \Gamma_a^*$ (see [AvSS, Section 2]), $\operatorname{Index}(P_\omega, \Gamma_a P_\omega \Gamma_a^*)$, is the finite integer given by

$$\operatorname{Index}(P_\omega, \Gamma_a P_\omega \Gamma_a^*) = \operatorname{tr}\left(P_\omega - \Gamma_a P_\omega \Gamma_a^*\right)^3. \tag{4.25}$$

Note that $\operatorname{Index}(P_\omega, \Gamma_a P_\omega \Gamma_a^*)$ is independent of $a \in \mathbb{Z}^{2*}$ [AvSS, Proposition 3.8], and hence it follows from the covariance relation (2.9) and properties of the index (use [AvSS, Proposition 2.4]) that for all $b \in \mathbb{Z}^2$ we have

$$\begin{aligned} \operatorname{Index}(P_{\tau_b\omega}, \Gamma_a P_{\tau_b\omega} \Gamma_a^*) &= \operatorname{Index}(U_b P_\omega U_b^*, \Gamma_a U_b P_\omega U_b^* \Gamma_a^*) \\ &= \operatorname{Index}(P_\omega, \Gamma_{a+b} P_\omega \Gamma_{a+b}^*) = \operatorname{Index}(P_\omega, \Gamma_a P_\omega \Gamma_a^*). \end{aligned} \tag{4.26}$$

Since $\operatorname{Index}(P_\omega, \Gamma_a P_\omega \Gamma_a^*)$ is a measurable function by (4.25), it follows from ergodicity that it must be constant almost surely (see [AvSS, Proposition 8.1]). In particular, this constant must be an integer, and, since constants are integrable,

$$\mathbb{E}\left\{\operatorname{Index}(P_\omega, \Gamma_a P_\omega \Gamma_a^*)\right\} = \operatorname{Index}(P_\omega, \Gamma_a P_\omega \Gamma_a^*) \quad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{4.27}$$

is an integer, and the lemma will follow if we show

$$\sigma_H(B, \lambda, E) = \mathbb{E}\left\{\operatorname{Index}(P_\omega, \Gamma_a P_\omega \Gamma_a^*)\right\}. \tag{4.28}$$

Let $T_\omega = P_\omega - \Gamma_a P_\omega \Gamma_a^*$. We have

$$\|T_\omega\|_q \leq \sum_{y \in \mathbb{Z}^2} \left\| \sum_{x \in \mathbb{Z}^2} \chi_{x+y} T_\omega \chi_x \right\|_q, \tag{4.29}$$

where

$$\begin{aligned}\Big\|\sum_{x\in\mathbb{Z}^2}\chi_{x+y}T_\omega\chi_x\Big\|_q^q &= \operatorname{tr}\Big|\sum_{x\in\mathbb{Z}^2}\chi_x T_\omega^*\chi_{x+y}T_\omega\chi_x\Big|^{\frac{q}{2}} \\ &= \sum_{x\in\mathbb{Z}^2}\operatorname{tr}|\chi_x T_\omega^*\chi_{x+y}T_\omega\chi_x|^{\frac{q}{2}} = \sum_{x\in\mathbb{Z}^2}\|\chi_{x+y}T_\omega\chi_x\|_q^q\,,\end{aligned} \tag{4.30}$$

and hence

$$\|T_\omega\|_q \le \sum_{y\in\mathbb{Z}^2}\Big(\sum_{x\in\mathbb{Z}^2}\|\chi_{x+y}T_\omega\chi_x\|_q^q\Big)^{\frac{1}{q}}\,, \tag{4.31}$$

which is the extension of [AG, Lemma 1] to the continuum. Note that if the right hand side of (4.31) is finite, then

$$T_\omega = \sum_{y\in\mathbb{Z}^2}\Big(\sum_{x\in\mathbb{Z}^2}\chi_{x+y}T_\omega\chi_x\Big) \quad \text{in } \mathcal{T}_q, \tag{4.32}$$

where $\mathcal{T}_q$ is the Banach space of compact operators with the norm $\|\ \|_q$, in the sense that for each $y\in\mathbb{Z}^2$ the series $\sum_{x\in\mathbb{Z}^2}\chi_{x+y}T_\omega\chi_x$ converges in $\mathcal{T}_q$, to, say, $T^{(y)}$ (but the series is not necessarily absolutely summable), the series $\sum_{y\in\mathbb{Z}^2}T^{(y)}$ converges absolutely in $\mathcal{T}_q$, and $T=\sum_{y\in\mathbb{Z}^2}T^{(y)}$.

It follows from (4.21) that

$$|||\chi_{x+y}T_\omega\chi_x|||_q \le 4\tfrac{|y|}{|x-a|}\,|||\chi_y P_\omega\chi_0|||_q\,, \tag{4.33}$$

and hence

$$\begin{aligned}\mathbb{E}\left(\|T_\omega\|_q\right) &\le \sum_{y\in\mathbb{Z}^2}\Big(\sum_{x\in\mathbb{Z}^2}|||\chi_{x+y}T_\omega\chi_x|||_q^q\Big)^{\frac{1}{q}} \\ &\le 4\Big(\sum_{x\in\mathbb{Z}^2}\tfrac{1}{|x-a|^q}\Big)^{\frac{1}{q}}\ell_q(B,\lambda,E)<\infty,\end{aligned} \tag{4.34}$$

where we used $q>2$. Since we also have $q\le 3$, and $\|S\|_r\le\|S\|_s$ for any $1\le s\le r<\infty$, we note that (4.24) follows from (4.34).

It remains to prove (4.28). To do so, note that it follows from (4.32) and (4.25) that

$$\operatorname{Index}(P_\omega,\Gamma_a P_\omega\Gamma_a^*) = \operatorname{tr}\,T_\omega^3 = \sum_{u,v\in\mathbb{Z}^2}\Big\{\sum_{x\in\mathbb{Z}^2}\operatorname{tr}\left(\chi_x T_\omega\chi_{x+u}T_\omega\chi_{x+v}T_\omega\chi_x\right)\Big\} \tag{4.35}$$

where the series in x is at first only known to be convergent for each u,v, but not absolutely convergent, to, say, $\zeta(u,v)$, and $\sum_{u,v\in\mathbb{Z}^2}|\zeta(u,v)|<\infty$.

To show that the series is actually absolutely convergent, we let r be given by $\frac{1}{r}+\frac{2}{q}=1$, so in particular $q<r$, and note that, using (4.21), we have

$$\sum_{u,v,x\in\mathbb{Z}^2} \mathbb{E}\left\{\operatorname{tr}\left|\chi_x T_\omega \chi_{x+u} T_\omega \chi_{x+v} T_\omega \chi_x\right|\right\} \tag{4.36}$$
$$\leq \sum_{u,v,x\in\mathbb{Z}^2} |||\chi_0 P_\omega \chi_u P_\omega \chi_v P_\omega \chi_0|||_1 \, \tfrac{4|u|}{|x-a|} \left\{ 2^{1-\frac{q}{r}} \left(\tfrac{4|u-v|}{|x+u-a|}\right)^{\frac{q}{r}} \right\} \tfrac{4|v|}{|x-a|}$$
$$\leq 64 \sum_{u,v\in\mathbb{Z}^2} |u||u-v|^{\frac{q}{r}}|v| \, |||\chi_0 P_\omega \chi_u P_\omega \chi_v P_\omega \chi_0|||_1 \sum_{a\in\mathbb{Z}^{2*}} \tfrac{1}{|a|^2|u-a|^{\frac{q}{r}}} < \infty,$$

since

$$\sum_{a\in\mathbb{Z}^{2*}} \tfrac{1}{|a|^2|u-a|^{\frac{q}{r}}} \leq \left(\sum_{a\in\mathbb{Z}^{2*}} \tfrac{1}{|a|^{\frac{6r}{3r-q}}}\right)^{\frac{3r-q}{3r}} \left(\sum_{a\in\mathbb{Z}^{2*}} \tfrac{1}{|a|^3}\right)^{\frac{q}{3r}} < \infty, \tag{4.37}$$

and

$$\sum_{u,v\in\mathbb{Z}^2} |u||u-v|^{\frac{q}{r}}|v| \, |||\chi_0 P_\omega \chi_u P_\omega \chi_v P_\omega \chi_0|||_1 \leq \left\{\sup_{x\in\mathbb{Z}^2} |x|^{\frac{q}{r}} \, |||\chi_x P_\omega \chi_0|||_r\right\} \{\ell_q(B,\lambda,E)\}^2$$
$$\leq \left\{\sup_{x\in\mathbb{Z}^2} |x| \, |||\chi_x P_\omega \chi_0|||_q\right\}^{\frac{q}{r}} \{\ell_q(B,\lambda,E)\}^2 \leq \{\ell_q(B,\lambda,E)\}^{2+\frac{q}{r}} < \infty. \tag{4.38}$$

We can thus take expectations in (4.35) obtaining

$$\mathbb{E}\left\{\operatorname{Index}(P_\omega, \Gamma_a P_\omega \Gamma_a^*)\right\} = \sum_{u,v\in\mathbb{Z}^2} \mathbb{E}\left\{\operatorname{tr}\left(\chi_0 P_\omega \chi_u P_\omega \chi_v P_\omega \chi_0\right)\right\} \times \tag{4.39}$$
$$\times \sum_{x\in\mathbb{Z}^2} (1-\gamma_a(x)\overline{\gamma_a}(x+u))(1-\gamma_a(x+u)\overline{\gamma_a}(x+v))(1-\gamma_a(x+v)\overline{\gamma_a}(x)).$$

On the other hand,

$$\sum_{x\in\mathbb{Z}^2} (1-\gamma_a(x)\overline{\gamma_a}(x+u))(1-\gamma_a(x+u)\overline{\gamma_a}(x+v))(1-\gamma_a(x+v)\overline{\gamma_a}(x)) \tag{4.40}$$
$$= \sum_{a\in\mathbb{Z}^{2*}} (1-\gamma_a(0)\overline{\gamma_a}(u))(1-\gamma_a(u)\overline{\gamma_a}(v))(1-\gamma_a(v)\overline{\gamma_a}(0)) = -2\pi i(u_1v_2-u_2v_1)$$

by Connes formula as in [AG, Appendix F] – see also [AG, Eqs. (4.14) and (5.1)].

Thus (4.28) follows from (4.39), (4.40), and (4.9). □

This completes the proof of Theorem 4.1.

5. Continuity of the Hall conductance

5.1. **Ergodic Landau Hamiltonians.** Theorem 2.2 follows immediately from the following theorem.

Theorem 5.1. *Let $H_{B,\lambda,\omega}$ be an ergodic Landau Hamiltonian. If for a given $(B,\lambda)\in(0,\infty)\times[0,\infty)$ the integrated density of states $N^{(B,\lambda)}(E)$ is continuous in E, then the Hall conductance $\sigma_H^{(B,\lambda)}(E)$ is continuous on $\Xi^{\{B,\lambda\}}_{(2,\infty)+}$. In particular, $\sigma_H^{(B,\lambda)}(E)$ is constant on each connected component of $\Xi^{\{B,\lambda\}}_{(2,3]+}$.*

To prove Theorem 5.1 we will use the following lemma.

Lemma 5.2. *Let $(B, E, \lambda) \in \Xi_{q+}$ with $q \in (2, \infty)$; set $\frac{1}{p} + \frac{2}{q} = 1$. Then there exists a neighborhood Φ of (B, E, λ) in Ξ, such that $\Phi \subset \Xi_{q+}$, and for all $(B', \lambda', E') \in \Phi$ we have, with $\sigma_H, \sigma'_H, P_\omega, P'_\omega$ for $\sigma_H(B, \lambda, E), \sigma_H(B', \lambda', E')$, $P_{B,\lambda,E,\omega}, P_{B',\lambda',E',\omega}$, respectively.*

$$|\sigma'_H - \sigma_H| \le C_{B,\lambda,E,q} \left\{ \sup_{u \in \mathbb{Z}^2} |||\chi_0 \left(P'_\omega - P_\omega\right) \chi_u|||_1^{\frac{1}{p}} \right\} \left\{\ell_{q+}(B, \lambda, E)\right\}^2 . \tag{5.1}$$

Proof. Given $(B, E, \lambda) \in \Xi_{q+}$ with $q \in (2, \infty)$, there exists a neighborhood Φ of (B, E, λ) in Ξ such that

$$\ell_q(B', \lambda', E') \le 2\ell_{q+}(B, \lambda, E) < \infty \tag{5.2}$$

for any $(B', \lambda', E') \in \Phi$. (It follows that $\Phi \subset \Xi_{q,+}$.) We write $\sigma_H, \sigma'_H, P_\omega, P'_\omega$ for $\sigma_H(B, \lambda, E), \sigma_H(B', \lambda', E')$, $P_{B,\lambda,E,\omega}, P_{B',\lambda',E',\omega}$, respectively. Using Lemma 4.2 and (4.7), we have

$$\begin{aligned} \frac{i}{2\pi}\left(\sigma'_H - \sigma_H\right) &= \mathbb{E}\left\{\mathrm{tr}\left\{\chi_0 \left(P'_\omega - P_\omega\right)\left[\left[P'_\omega, \hat{X}_1\right], \left[P'_\omega, \hat{X}_2\right]\right]\chi_0\right\}\right\} \\ &\quad + \mathbb{E}\left\{\mathrm{tr}\left\{\chi_0 P_\omega\left[\left[\left(P'_\omega - P_\omega\right), \hat{X}_1\right], \left[P'_\omega, \hat{X}_2\right]\right]\chi_0\right\}\right\} \\ &\quad + \mathbb{E}\left\{\mathrm{tr}\left\{\chi_0 P_\omega\left[\left[P_\omega, \hat{X}_1\right], \left[\left(P'_\omega - P_\omega\right), \hat{X}_2\right]\right]\chi_0\right\}\right\} \\ &\equiv \sigma_1 + \sigma_2 + \sigma_3, \end{aligned} \tag{5.3}$$

where $\sigma_1, \sigma_2, \sigma_3$ can be shown to be well defined as in the proof of Lemma 4.2, and can be written similarly to (4.9). Thus, with $\frac{1}{p} + \frac{2}{q} = 1$, where $p < \infty$ since $q > 2$, we have

$$\begin{aligned} |\sigma_1| &\le \sum_{u,v \in \mathbb{Z}^2} |(u_1 - v_1)v_2 - (u_2 - v_2)v_1| \mathbb{E}\left\{\mathrm{tr}\left|\chi_0\left(P'_\omega - P_\omega\right)\chi_u P'_\omega \chi_v P'_\omega \chi_0\right|\right\} \\ &\le 8 \left\{\sup_{u \in \mathbb{Z}^2} |||\chi_0\left(P'_\omega - P_\omega\right)\chi_u|||_p\right\}\left\{\ell_{q+}(B, \lambda, E)\right\}^2 \\ &\le 16 \left\{\sup_{u \in \mathbb{Z}^2} |||\chi_0\left(P'_\omega - P_\omega\right)\chi_u|||_1^{\frac{1}{p}}\right\}\left\{\ell_{q+}(B, \lambda, E)\right\}^2, \end{aligned} \tag{5.4}$$

with similar estimates for $|\sigma_2|$ and $|\sigma_3|$. The desired estimate (5.1) now follows from (5.3) and (5.4). □

Proof of Theorem 5.1. In view of Theorem 4.1, it suffices to show that if for a given $(B, \lambda) \in (0, \infty) \times [0, \infty)$ the integrated density of states $N^{(B,\lambda)}(E)$ is continuous in E, then the Hall conductance $\sigma_H^{(B,\lambda)}(E)$ is continuous on $\Xi^{\{B,\lambda\}}_{(2,\infty)+}$. This follows immediately from Lemma 5.2, since for $E_1 \le E_2$ we have, for all $u \in \mathbb{Z}^2$,

$$\begin{aligned} |||\chi_0\left(P_{B,\lambda,E_2,\omega} - P_{B,\lambda,E_1,\omega}\right)\chi_u|||_1 &\le |||\chi_0\left(P_{B,\lambda,E_2,\omega} - P_{B,\lambda,E_1,\omega}\right)\chi_0|||_1 \\ &= N^{(B,\lambda)}(E_2) - N^{(B,\lambda)}(E_1). \end{aligned} \tag{5.5}$$

□

5.2. **The Anderson-Landau Hamiltonian.** Theorem 2.4 follows from the following theorem.

Theorem 5.3. *Let $H^{(A)}_{B,\lambda,\omega}$ be the Anderson-Landau Hamiltonian. Then the Hall conductance $\sigma_H(B,\lambda,E)$ is defined on $\Xi_{[2,\infty)}$, integer valued on $\Xi_{(2,3]}$, and Hölder-continuous on $\Xi_{(2,\infty)+}$. In particular, $\sigma_H(B,\lambda,E)$ is constant on each connected component of $\Xi_{(2,3]+}$.*

In view of Theorems 4.1 and 5.1, all that remains to finish the proof of Theorem 5.3 is to show that for the Anderson-Landau Hamiltonian the Hall conductance $\sigma_H(B,\lambda,E)$ is Hölder-continuous on $\Xi_{(2,\infty)+}$. This will follow from Lemma 5.2 and the following lemma, which improves on a result of Combes, Hislop, Klopp, and Raikov [CoHKR]: the integrated density of states of the Anderson-Landau Hamiltonian $N(B,\lambda,E)$ is jointly Hölder continuous in (B,E) for $\lambda>0$. More precisely, they proved that given given $\lambda>0$, $\alpha,\delta\in(0,1)$, and a compact set $Y\subset(0,\infty]\times\mathbb{R}$, there exists a constant $C_{Y,\alpha,\delta}(\lambda)$ such that

$$|N(B',\lambda,E')-N(B,\lambda,E)|\le C_{Y,\alpha,\delta}(\lambda)\left(|B'-B|^{\frac{\alpha}{4}}+|E'-E|^{\delta}\right) \tag{5.6}$$

for all $(B,E),(B',E')\in Y$, and the constant $C_{Y,\alpha,\delta}(\lambda)$ is locally bounded for$\lambda>0$. (Although the fact that $C_{Y,\alpha,\delta}(\lambda)$ is locally bounded is not explicitly stated in [CoHKR], it is implicit in the proof.)] Hölder continuity in the energy was previously known in special cases [CoH, W, HuLMW2, CoHK1]. We strengthen this result, proving joint Hölder-continuity of $\chi_0 P_{B,\lambda,E,\omega}\chi_0$ in the $|||\ |||_1$ norm with respect to (B,E,λ).

Lemma 5.4. *Let $H^{(A)}_{B,\lambda,\omega}$ be the Anderson-Landau Hamiltonian. Fix $\alpha,\delta,\eta\in(0,1)$. Then, given a compact subset K of Ξ, there exists a constant $C_{K,\alpha,\delta,\eta}$ such that*

$$\begin{aligned}\sup_{u\in\mathbb{Z}^2}&|||\chi_0\left(P_{B',\lambda',E',\omega}-P_{B'',\lambda'',E'',\omega}\right)\chi_u|||_1\\ &\le C_{K,\alpha,\delta,\eta}\left(|B'-B|^{\frac{\alpha}{5}}+|E'-E''|^{\delta}+|\lambda'-\lambda''|^{\frac{\eta}{3}}\right)\end{aligned} \tag{5.7}$$

for all $(B',\lambda',E'),(B'',\lambda'',E'')\in K$.

Lemma 5.4 will follow from the above stated result of [CoHKR] and Lemma 5.5 below. Note that if $E''\le E'$ we have $P_{B,\lambda,E',\omega}-P_{B,\lambda,E'',\omega}\ge 0$, so the hypothesis of Lemma 5.5 follow from (5.6).

Lemma 5.5. *Let $H^{(A)}_{B,\lambda,\omega}$ be the Anderson-Landau Hamiltonian. Let $\delta\in(0,1)$. Suppose that for every bounded interval I and $(B,\lambda)\in(0,\infty)^2$ there exists a constant $C_I(B,\lambda)$, locally bounded in (B,λ), such that for all $E',E''\in I$ we have*

$$|||\chi_0\left(P_{B,\lambda,E',\omega}-P_{B,\lambda,E'',\omega}\right)\chi_0|||_1\le C_I(B,\lambda)|E'-E''|^{\delta}. \tag{5.8}$$

Given $K=[B_1,B_2]\times[\lambda_1,\lambda_2]\times[E_1,E_2]\subset\Xi$, there is a constant C_K, such that for all $E\in[E_1,E_2]$ and $u\in\mathbb{Z}^2$ we have

$$|||\chi_0\left(P_{B,\lambda',E,\omega}-P_{B,\lambda'',E,\omega}\right)\chi_u|||_1\le C_K|\lambda'-\lambda''|^{\frac{\delta}{\delta+2}}, \tag{5.9}$$

for all $B\in[B_1,B_2]$ and $\lambda',\lambda''\in[\lambda_1,\lambda_2]$, and

$$|||\chi_0\left(P_{B',\lambda,E,\omega}-P_{B'',\lambda,E,\omega}\right)\chi_u|||_1\le C_K|B'-B''|^{\frac{\delta}{\delta+4}}, \tag{5.10}$$

for all $B',B''\in[B_1,B_2]$ and $\lambda\in[\lambda_1,\lambda_2]$.

Proof. It suffices to consider the case when $B_2-B_1<1$ and $\lambda_2-\lambda_1<1$, We set $I=[E_1-1,E_2]$. Note that (5.8) holds for $(B,\lambda)\in[B_1,B_2]\times[\lambda_1,\lambda_2]$ and $E',E''\in I$ with $C_I\equiv\sup_{(B,\lambda)\in[B_1,B_2]\times[\lambda_1,\lambda_2]}C_I(B,\lambda)<\infty$. (This includes the

case $\lambda_1 = 0$ with a slightly modified interval I, although this case is not included in the hypothesis (5.8). The reason is that since $K \subset \Xi$, if $\lambda_1 = 0$ the interval $[E_1, E_2]$ cannot contain any Landau level for $B \in [B_1, B_2]$. In this case we set $I = [E_1 - \rho, E_2]$, where $0 < \rho \le 1$ is chosen so I also does not contain a Landau level for some $B \in [B_1, B_2]$. The proof applies also in this case except that we take $B_2 - B_1 < \rho$ and $\lambda_2 - \lambda_1 < \rho$.)

We fix a function $f \in C^\infty(\mathbb{R})$, such that $0 \le f(t) \le 1$, $f(t) = 1$ if $t \le 0$, and $f(t) = 0$ if $t \ge 1$.

We prove (5.9) first. Let $E \in [E_1, E_2]$, $B \in [B_1, B_2]$, and $\lambda', \lambda'' \in [\lambda_1, \lambda_2]$. We let $\gamma = |\lambda' - \lambda''|^\alpha$, where $\alpha \in (0, 1)$ will be chosen later. We set $g(t) = f\left(\frac{t-(E-\gamma)}{\gamma}\right)$; note $g \in C^\infty(\mathbb{R})$, with $0 \le g(t) \le 1$, $g(t) = 1$ if $t \le E - \gamma$, $g(t) = 0$ if $t \ge E$. We write

$$\begin{aligned} P_{B,\lambda',E,\omega} - P_{B,\lambda'',E,\omega} = \{P_{B,\lambda',E,\omega} - g(H_{B,\lambda',\omega})\} \\ + \{g(H_{B,\lambda',\omega}) - g(H_{B,\lambda'',\omega})\} + \{g(H_{B,\lambda'',\omega}) - P_{B,\lambda'',E,\omega}\}. \end{aligned} \tag{5.11}$$

By construction, for any $\lambda \ge 0$ we have

$$0 \le P_{B,\lambda,E,\omega} - g(H_{B,\lambda,\omega}) \le P_{B,\lambda,E,\omega} - P_{B,\lambda,E-\gamma,\omega}, \tag{5.12}$$

and thus, for $\lambda^\# = \lambda', \lambda''$ and any $u \in \mathbb{Z}^2$, we have

$$\begin{aligned} &|||\chi_0 \left(P_{B,\lambda^\#,E,\omega} - g(H_{B,\lambda^\#,\omega})\right) \chi_u|||_1 \\ &\le \left|\left|\left|\chi_0 \left(P_{B,\lambda^\#,E,\omega} - g(H_{B,\lambda^\#,\omega})\right)^{\frac{1}{2}}\right|\right|\right|_2 \left|\left|\left|\left(P_{B,\lambda^\#,E,\omega} - g(H_{B,\lambda^\#,\omega})\right)^{\frac{1}{2}} \chi_u\right|\right|\right|_2 \\ &= |||\chi_0 \left(P_{B,\lambda^\#,E,\omega} - g(H_{B,\lambda^\#,\omega})\right) \chi_0|||_1 \\ &\le |||\chi_0 \left(P_{B,\lambda^\#,E,\omega} - P_{B,\lambda^\#,E-\gamma,\omega}\right) \chi_0|||_1 \le C_I \gamma^\delta. \end{aligned} \tag{5.13}$$

We now estimate the middle term in the right hand side of (5.11). Let $R_{B,\lambda,B\omega}(z) = (H_{B,\lambda,\omega} - z)^{-1}$ be the resolvent. Recall (e.g., [BoGKS]) that

$$\|\chi_v R_{\lambda,B,\omega}(z)\|_2 \le c_\lambda \frac{1 + |z|}{\mathrm{Im} z}, \tag{5.14}$$

with a constant c_λ independent of B, $v \in \mathbb{Z}^2$, and ω, and locally bounded in λ. The Helffer-Sjöstrand formula with a quasi analytic extension of g of order 3 (e.g., [D]), combined with the resolvent equation and (5.14), yields

$$|||\chi_0 \left(g(H_{B,\lambda',\omega}) - g(H_{B,\lambda'',\omega})\right) \chi_u|||_1 \le C \frac{|\lambda' - \lambda''|}{\gamma^2}, \tag{5.15}$$

where the constant C depends only on $E_1, E_2, \lambda_1, \lambda_2$, our choice of the function f, and fixed parameters.

Thus, combining (5.11), (5.13), and (5.15). we get

$$\begin{aligned} |||\chi_0 \left(P_{\lambda',E',\omega} - P_{\lambda'',E'',\omega}\right) \chi_u|||_1 &\le 2C_I \gamma^\delta + C \frac{|\lambda' - \lambda''|}{\gamma^2} \\ &= 2C_I |\lambda' - \lambda''|^{\alpha\delta} + C|\lambda' - \lambda''|^{1-2\alpha} = (2C_I + C)|\lambda' - \lambda''|^{\frac{\delta}{\delta+2}}, \end{aligned} \tag{5.16}$$

where we chose $\alpha = \frac{1}{\delta+2}$ to optimize the bound.

To prove (5.10), we start by repeating the above proof varying B instead of λ. The only difference is in the equivalent of the estimate (5.15). Here we use [CoHKR,

Proposition 5.1], observing that its proof (note [CoHKR, Eqs. (5.12) and (5.13)]) actually proves the stronger result

$$|||\chi_0 \left(g(H_{B',\lambda,\omega}) - g(H_{B'',\lambda,\omega})\right) \chi_u|||_1 \leq \tilde{C} \frac{|B' - B''|}{\gamma^4}, \tag{5.17}$$

where now $\gamma = |B' - B''|^\alpha$, and the constant $\tilde{C}$ depends only on $E_1, E_2, \lambda_1, \lambda_2, B_1, B_2$, our choice of the function f, and fixed parameters. Proceeding as before, we see that in this case we should choose $\alpha = \frac{1}{\delta+4}$, in which case we get (5.10). □

6. Delocalization for ergodic Landau Hamiltonians with open gaps

We now prove Corollary 2.3 by proving the following theorem

Theorem 6.1. *Let $H_{B,\lambda,\omega}$ be an ergodic Landau Hamiltonian. Suppose the integrated density of states $N^{(B,\lambda)}(E)$ is continuous in E for all $(B,\lambda) \in (0,\infty)\times[0,\infty)$ satisfying the disjoint bands condition (2.31). Then for all such (B,λ) the "localization length" $\ell^{\{B,\lambda\}}_{(2,3]+}$ diverges near each Landau level: for each $n = 1, 2, \ldots$ there exists an energy $E_n(B,\lambda) \in \mathcal{B}_n(B,\lambda)$ such that*

$$\ell^{\{B,\lambda\}}_{(2,3]+}(E_n(B,\lambda)) = \infty. \tag{6.1}$$

We start th eproof of Theorem 6.1 by setting, for $n = 1, 2, \ldots$,

$$\mathbb{G}_n = \left\{(B,\lambda,E) \in \Xi;\ \lambda(M_1 + M_2) < 2B,\ E \in (B_{n-1} + \lambda M_2, B_n - \lambda M_1)\right\}. \tag{6.2}$$

In view of (2.12) and (2.30), we have

$$\bigcup_{n=1}^{\infty} \mathbb{G}_n = \Xi \setminus \bigcup_{B\in(0,\infty)} \bigcup_{\lambda\in[0,\infty)} \bigcup_{n=1}^{\infty} \{(B,\lambda)\} \times \mathcal{B}_n(B,\lambda) \subset \Xi_{\mathrm{NS}} \subset \Xi_{(2,3]+}. \tag{6.3}$$

It is well known that $\sigma_H(B,0,E) = n$ if $E \in]B_n, B_{n+1}[$ for all $n = 0, 1, 2\ldots$ [AvSS, BeES]. Given $n \in \mathbb{N}$ and $(B,\lambda_1,E) \in \mathbb{G}_n$, we can find $\lambda_E > \lambda_1$ such that $E \in \mathbb{G}_n^{(B,\lambda)}$ for all $\lambda \in I = [0, \lambda_E[$. It follows that, with probability one,

$$P_\lambda = -\tfrac{1}{2\pi i} \int_\Gamma R_\lambda(z)\, \mathrm{d}z \quad \text{for all } \lambda \in I, \tag{6.4}$$

where $P_\lambda = P_{B,\lambda,E,\omega}$, $R_\lambda(z) = (H_{B,\lambda,\omega} - z)^{-1}$, and Γ is a bounded contour such that $\mathrm{dist}(\Gamma, \sigma(H_{B,\lambda,\omega})) \geq \eta > 0$ for all $\lambda \in I$. (Note $H_{B,\lambda,\omega} \geq B - \lambda_E M_1$ for all $\lambda \in I$.) It follows that there is a constant K such that (cf. [BoGKS, Proposition 2.1])

$$\|R_\lambda(z)\chi_x\|_2 \leq K \quad \text{for all } x \in \mathbb{Z}^2, z \in \Gamma, \lambda \in I. \tag{6.5}$$

Given $\lambda, \xi \in I$, it follows from (6.4) and the resolvent identity that

$$Q_{\lambda,\xi} := P_\xi - P_\lambda = \tfrac{(\xi-\lambda)}{2\pi i} \int_\Gamma R_\lambda(z) V R_\xi(z)\, \mathrm{d}z, \tag{6.6}$$

with $V = V_\omega$ (recall $\|V\| \leq \widetilde{M} := \max\{M_1, M_2\}$). Letting $\sigma_\lambda = \sigma_H(B,\lambda,E)$, it follows from Lemma 5.2 that for all $\lambda \in I$, taking $\xi \in I$ in a suitable neighborhood of λ, we have

$$|\sigma_\lambda - \sigma_\xi| \leq C'_{B,\lambda,E} \left\{ \sup_{u\in\mathbb{Z}^2} |||\chi_0 Q_{\lambda,\xi} \chi_u|||_1^{\frac{1}{3}} \right\} \leq C'_{B,\lambda,E} \left\{ \tfrac{|\xi-\lambda|}{2\pi} \widetilde{M}\, |\Gamma|\, K^2 \right\}^{\frac{1}{3}}, \tag{6.7}$$

so σ_λ is a continuous function of λ in the interval I. By Theorem 4.1, σ_λ is constant in I, and hence we conclude that

$$\sigma_H(B,\lambda,E) = \sigma_H(B,0,E) = n \quad \text{for all} \quad (B,\lambda,E) \in \mathbb{G}_n. \tag{6.8}$$

Now, let (B,λ) satisfy (2.31), and suppose $\mathcal{B}_n(B,\lambda) \subset \Xi^{\{B,\lambda\}}_{(2,3]+}$ for some $n \in \mathbb{N}$. We then have

$$(B_{n-1} + \lambda M_1, B_{n+1} - \lambda M_2) = \mathbb{G}^{(B,\lambda)}_{n-1} \cup \mathcal{B}_n(B,\lambda) \cup \mathbb{G}^{(B,\lambda)}_n \subset \Xi^{\{B,\lambda\}}_{(2,3]+}. \tag{6.9}$$

Since the integrated density of states $N^{(B,\lambda)}(E)$ is assumed to be continuous in E, it follows from Theorem 5.1 that the Hall conductance $\sigma_H(B,\lambda,E)$ is constant on the interval $(B_{n-1} + \lambda M_1, B_{n+1} - \lambda M_2)$,and hence has the same value on the spectral gaps $\mathbb{G}^{(B,\lambda)}_{n-1}$ and $\mathbb{G}^{(B,\lambda)}_n$, which contradicts (6.8). Thus we conclude that $\mathcal{B}_n(B,\lambda)$ cannot be a subset of $\Xi^{\{B,\lambda\}}_{(2,3]+}$, which proves Theorem 6.1.

7. Dynamical delocalization for the Anderson-Landau Hamiltonian with closed gaps

In this section we prove Theorem 2.5.

Let $H^{(A)}_{B,\lambda,\omega}$ be an Anderson-Landau Hamiltonian as in (2.5)-(2.6), with a common probability distribution μ with $\operatorname{supp}\mu = [-M_1, M_2]$ with $M_1, M_2 \in (0,\infty)$. As shown in Appendix B, we have

$$\Sigma_{B,\lambda} = \bigcup_{n\in\mathbb{N}} I_n(B,\lambda), \quad \text{where} \quad I_n(B,\lambda) = [E_-(n,B,\lambda), E_+(n,B,\lambda)], \tag{7.1}$$

where, for all $B > 0$ and $n \in \mathbb{N}$, $\pm E_\pm(n,B,\lambda)$ are increasing, continuous functions of $\lambda > 0$, depending on u and M_1, M_2, but not on other details of the measure μ. We set $E_+(0,B,\lambda) = -\infty$. We have

$$B_{n-1} \le E_-(n,B,\lambda) < B_n < E_+(n,B,\lambda) \le B_{n+1} \quad \text{for all} \quad n \in \mathbb{N}, \tag{7.2}$$
$$B - \lambda M_1 \le E_-(1,B,\lambda) = E_0(B,\lambda) := \inf \Sigma_{B,\lambda} < B,$$

(Note that $B - \lambda M_1 \le E_0(B,\lambda)$ follows from (2.12).) In

If (2.31) holds, then $E_+(n,B,\lambda) < E_-(n+1,B,\lambda)$ for all $n \in \mathbb{N}$ and the spectral gaps do not close. If for some $n \in \mathbb{N}$ we have $E_+(n,B,\lambda) \ge E_-(n+1,B,\lambda)$, the n-th spectral gap (B_n, B_{n+1}) has closed, i.e., $[B_n, B_{n+1}] \subset \Sigma_{B,\lambda}$.

Let us now assume that the single-site potential u in (2.6) satisfies

$$0 < U_- \le U(x) := \sum_{i\in\mathbb{Z}^2} u(x-i) \le 1, \tag{7.3}$$

for some constant U_-. (The upper bound is simply a normalization we had already assumed.) Then, as shown in Appendix B, we have

$$B_n + \lambda M_2 U_- \le E_+(n,B,\lambda) \quad \text{for} \quad \lambda \in \left(0, \tfrac{2B}{M_2 U_-}\right), \tag{7.4}$$

$$B_n - \lambda M_1 U_- \ge E_-(n,B,\lambda) \quad \text{for} \quad \lambda \in \left(0, \tfrac{2B}{M_1 U_-}\right), \tag{7.5}$$

$$B - \lambda M_1 U_- \ge E_-(1,B,\lambda) = E_0(B,\lambda) \quad \text{for all} \quad \lambda \ge 0. \tag{7.6}$$

It follows that if

$$\lambda(M_1 + M_2) U_- \ge 2B, \tag{7.7}$$

all the internal spectral gaps close, i.e.,

$$\Sigma_{B,\lambda} = [E_0(B,\lambda), \infty). \tag{7.8}$$

Theorem 2.5(i) is proven.

To prove Theorem 2.5(ii), we assume (2.41) and fix $\widehat{\lambda} > \frac{1}{U_-}B$, and $\delta \in (0, B)$. Let $J_n(B)$ be as in (2.44), we set

$$\begin{aligned} \widehat{J}_n(B) &:= \left(B_n + \tfrac{\delta}{2}, B_{n+1} - \tfrac{\delta}{2}\right), \quad n \in \mathbb{N}, \\ \widehat{J}_0(B) &:= \left(-\infty, B - \tfrac{\delta}{2}\right) \subset (-\infty, B). \end{aligned} \tag{7.9}$$

We will prove (2.45) by a multiscale analysis. The multiscale analysis is carried on for the finite volume operators defined in [GKS, Section 4 and 5]; the Anderson-Landau Hamiltonian satisfies all the requirements for the multiscale analysis plus a Wegner estimate [GKS, Sections 4 and 5]. We take scales $L \in L_B\mathbb{N}$, where $L_B \geq 1$ is defined in [GKS, Eq. (5.1)], and consider boxes $\Lambda_L(x) = x + [-\frac{L}{2}, \frac{L}{2})^2$, $x \in \mathbb{R}^2$, and let $\widetilde{\Lambda}_L(x) = \Lambda_L(x) \cap \mathbb{Z}^2$. We define finite volume operators $H_{B,\lambda,0,L,\omega}$ on $\mathrm{L}^2(\Lambda_L(0))$ as in [GKS, Eq. (5.2)]:

$$\begin{aligned} H_{B,\lambda,0,L,\omega} &= H_{B,0,L} + \lambda V_{0,L,\omega} \quad \text{on} \quad \mathrm{L}^2(\Lambda_L(0)), \\ V_{0,L,\omega}(x) &= \sum_{i \in \widetilde{\Lambda}_{L-\delta_u}(0)} \omega_i \, u(x-i), \end{aligned} \tag{7.10}$$

where $H_{B,0,L}$ is defined in [GKS, Sections 5] and $\operatorname{supp} u \subset \left(-\frac{\delta_u}{2}, \frac{\delta_u}{2}\right)^2$, and then define $H_{B,\lambda,\omega,x,L}$ for all $x \in \mathbb{Z}^2$ by [GKS, Eq. (4.3)]. (We prescribed periodic boundary condition for the (free) Landau Hamiltonian at the square centered at 0, and used the magnetic translations to define the finite volume operators in all other squares by [GKS, Eq. (4.3)]; in the square centered at $x \in \mathbb{Z}^2$ the potential $V_{x,L,\omega}$ is exactly as in (7.10) except that the sum is now over $i \in \widetilde{\Lambda}_{L-\delta_u}(x)$.)

A Wegner estimate is given in [GKS, Theorem 5.1] and extended in [CoHK2, Theorem 4.3]; note that the constants in the Wegner estimate can be chosen uniformly in $\lambda \in [\lambda_1.\lambda_2]$ if $\lambda_1 > 0$. It follows that for a closed interval $I \subset (B_n, B_{n+1})$, $n = 0, 1, 2, \ldots$, they can be chosen uniformly in $\lambda \in [0, \widehat{\lambda}]$. (But note that the constants will depend on the interval I, and hence for $I = \widehat{J}_n(B)$ they will depend on n.) But one has to be careful in the multiscale analysis, since $\|\rho\|_\infty$ appears in the Wegner estimate, (2.41) gives $\|\rho\|_\infty = \frac{\eta+1}{2}$, and we will prove (2.45) for η sufficiently large.

All these issues can be taken in consideration by applying the finite volume criterion for localization given in [GK2, Theorem 2.4], in a similar way to the application in [GK2, Proof of Theorem 3.1].

We write $\Lambda = \Lambda_L(x)$, $H_{B,\lambda,L,\omega} = H_{B,\lambda,x,L,\omega}$, etc. If $\lambda\,|\omega_i| \leq \frac{\delta}{2}$ for all $i \in \widetilde{\Lambda}$, then we have by Lemma A.1 (it also applies to finite volume operators) that

$$\sigma\left(H_{B,\lambda,L,\omega}\right) \subset \bigcup_{n=1}^{\infty} \left[B_n - \tfrac{\delta}{2}, B_n + \tfrac{\delta}{2}\right]. \tag{7.11}$$

We have

$$\begin{aligned} \inf_{\lambda \in [0,\widehat{\lambda}]} \mathbb{P}\left\{\lambda\,|\omega_i| \leq \tfrac{\delta}{2} \quad \text{for all } i \in \widetilde{\Lambda}\right\} &\geq 1 - L^2\, \mathbb{P}\left\{\widehat{\lambda}\,|\omega_0| > \tfrac{\delta}{2}\right\} \\ &= 1 - L^2 \left(1 - \tfrac{\delta}{2\widehat{\lambda}}\right)^{\eta} \end{aligned} \tag{7.12}$$

where $\frac{\delta}{2\widehat{\lambda}} < \frac{U_-}{2} \leq \frac{1}{2}$

Given ω satisfying (7.11), $E \in J_n(B)$ implies $\operatorname{dist}(E, \sigma(H_{B,\lambda,L,\omega})) > \frac{\delta}{2}$. Let $R_{B,\lambda,L,\omega}(E) = (H_{B,\lambda,L,\omega} - E)^{-1}$. It follows from the Combes estimate (cf. [GK1, Theorem 1]; note that the estimate holds for finite volume operators with periodic boundary condition with uniform constants for large enough volumes using the distance on the torus, cf. [FK2, Lemma 18] and [KlK1, Theorem 3.6]) that

$$\|\chi_x R_{B,\lambda,L,\omega}(E)\chi_y\| \leq \tfrac{C_1}{\delta}\mathrm{e}^{-C_2\delta L} \quad \text{for all } x, y \in \widetilde{\Lambda} \text{ with } |x-y| \geq \tfrac{L}{10}, \tag{7.13}$$

where $C_1, C_2 > 0$ are constants, depending only on n, B, u.

Let us fix $n \in \mathbb{N}$ and prove that $J_n(B) \subset \Xi_{\mathrm{DL}}^{(B,\lambda)}$ for all $\lambda \in [0, \widehat{\lambda}]$. (The case $n = 0$ can be handled in a similar manner.) We take the constants in the Wegner estimate valid for subintervals of $\widehat{J}_n(B)$, uniformly in $\lambda \in [0, \widehat{\lambda}]$. Thus, if we have (7.11), we will have the condition whose probability is estimated in [GKS, Eq. (2.17)] if

$$L^9 \tfrac{C_1}{\delta}\mathrm{e}^{-C_2\delta L} < \frac{C_3}{\eta + 1}, \tag{7.14}$$

where C_3 is another constant depending only on n, B, u, and δ.

We now take $L_0(n)$ satisfying [GKS, Eq. (2.16)] and large enough for the Wegner estimate, and for $L_0 \geq L_0(n)$ we set

$$\eta(n, L_0) = 1 + \tfrac{C_3\delta}{2C_1} L_0^{-9}\mathrm{e}^{C_2\delta L_0}, \tag{7.15}$$

so (7.14) holds with $L = L_0$ and $\eta = \eta(n, L_0)$. Since

$$\lim_{L_0\to\infty} L_0^2 \left(1 - \tfrac{\delta}{2\widehat{\lambda}}\right)^{\eta(n,L_0)} = 0 \tag{7.16}$$

Thus we can find $\eta(n) > 0$ such that for all $\eta \geq \eta(n)$ there exists $L_0(\eta) \geq L_0(n)$ for which we have [GKS, Eq. (2.17)], so $E \in J_n(B)$ implies $E \in \Xi_{\mathrm{DL}}^{(B,\lambda)}$.

Thus given $N \in \mathbb{N}$, letting $\eta_N = \max_{n=0,1,2,\ldots,N} \eta(n)$, we have (2.45) for $\eta \geq \eta_N$.

Since the Hall conductance $\sigma_H(B, 0, E) = n$ if $E \in (B_n, B_{n+1})$ for all $n = 0, 1, 2\ldots$ [AvSS, BeES], it follows from Theorem 2.4 that for $\eta \geq \eta_N$ we have

$$\sigma_H(B, \lambda, E) = n \quad \text{for all} \quad (\lambda, E) \in [0, \widehat{\lambda}] \times J_n(B). \tag{7.17}$$

We now proceed as in [GKS, Proof of Theorem 2.2], using again Theorem 2.4 (here we could also use Theorem 2.2), to conclude that for $n = 1, 2, \ldots, N$ we have $E_n(B, \lambda) \in [B_n - \delta, B_n + \delta]$ with $L_+^{\{B,\lambda\}}(E_n(B, \lambda)) = \infty$, so we have (2.46), and (2.47) follows from [GK3, Theorem 2.11], as in [GKS, Theorem 2.2].

Theorem 2.5 is proven.

Appendix A. The spectrum of Landau Hamiltonians with bounded potentials

In the appendix we justify (2.12).

Lemma A.1. *Let $H = H_B + W$, where H_B is the free Landau Hamiltonian as in (2.2), and $-M_1 \leq W \leq M_2$, where $M_1, M_2 \in [0, \infty)$. Then*

$$\sigma(H) \subset \bigcup_{n=1}^{\infty} [B_n - M_1, B_n + M_2]. \tag{A.1}$$

Proof. THe lemma follows from [K, Theorem V.4.10] by writing

$$H = \left(H_B - \tfrac{M_1 - M_2}{2}\right) + \left(W + \tfrac{M_1 - M_2}{2}\right). \tag{A.2}$$

□

Appendix B. The spectrum of Anderson-Landau Hamiltonians

Consider an Anderson-Landau Hamiltonian $H_{B,\lambda,\omega} = H^{(A)}_{B,\lambda,\omega}$ as in (2.5)-(2.6), and suppose that

$$\operatorname{supp} \mu = [-M_1, M_2] \quad \text{with } M_1, M_2 \in (0,\infty). \tag{B.1}$$

(The argument applies also to the case $M_1, M_2 \in [0,\infty)$ with $M_1 + M_2 > 0$, with the obvious modifications.) In this appendix we make no other hypotheses on the common probability distribution μ. It follows from [KiM2, Theorem 4], which applies also to Anderson-Landau Hamiltonians, that under these hypotheses we have

$$\Sigma_{B,\lambda} = \bigcup_{\omega \in \Omega_{\text{supp}}} \sigma\left(H_{B,\lambda,\omega}\right), \quad \text{where} \quad \Omega_{\text{supp}} := [-M_1, M_2]^{\mathbb{Z}^2}. \tag{B.2}$$

We consider squares $\Lambda_L := [-\frac{L}{2}, \frac{L}{2})$ centered at the origin with side $L > 0$. Given such a square Λ, we define $\omega^{(\Lambda)}$ by $\omega^{(\Lambda)}_j = \omega_j$ if $j \in \Lambda$ and $\omega^{(\Lambda)}_j = 0$ otherwise, and set

$$H^{(\Lambda)}_{B,\lambda,\omega} := H_B + \lambda V^{(\Lambda)}_\omega, \quad \text{where} \quad V^{(\Lambda)}_\omega = V_{\omega^{(\Lambda)}}. \tag{B.3}$$

Note that $V^{(\Lambda)}_\omega$ is relatively compact with respect to H_B, so Σ_B is also the essential spectrum of $H^{(\Lambda)}_{B,\lambda,\omega}$. In particular, $H^{(\Lambda)}_{B,\lambda,\omega}$ has discrete spectrum in the spectral gaps $\{\mathcal{G}_n(B) := (B_n, B_{n+1}),\ n = 0, 1, \ldots\}$ of H_B. Since $\omega^{(\Lambda)} \in \Omega_{\text{supp}}$ if $\omega \in \Omega_{\text{supp}}$, it follows that

$$\Sigma_B \subset \Sigma_{B,\lambda} = \overline{\bigcup_{n=1}^{\infty} \bigcup_{\omega \in \Omega_{\text{supp}}} \sigma\left(H^{(\Lambda_{L_n})}_{B,\lambda,\omega}\right)}, \tag{B.4}$$

for any $L_n \to \infty$. (This uses (B.2) plus the fact that $H^{(\Lambda_{L_n})}_{B,\lambda,\omega}$ converges to $H_{B,\lambda,\omega}$ in the strong resolvent sense.) In particular, it follows from (B.1) that $\Sigma_{B,\lambda}$ is increasing with λ.

Let $\omega \in \Omega_{\text{supp}}$, $\omega^{(\Lambda)} > 0$, that is, $\omega_j \geq 0$ for all $j \in \Lambda$ and $\sum_{j\in\Lambda} \omega_j > 0$. In this case $V^{(\Lambda)}_\omega \geq 0$, and

$$\Sigma_B \subset \sigma\left(H^{(\Lambda)}_{B,\lambda,\omega}\right) \subset \bigcup_{n=1}^{\infty} [B_n, B_n + \lambda M_2]. \tag{B.5}$$

We now use a modified Birman-Schwinger method, following [FK4, Section 4]. We fix $n \in \mathbb{N}$ and set

$$\mathcal{R}(E) = -\sqrt{V^{(\Lambda)}_\omega}\,(H_B - E)^{-1}\sqrt{V^{(\Lambda)}_\omega} \quad \text{for} \quad E \in (B_n, B_{n+1}), \tag{B.6}$$

a compact self-adjoint operator. Let $r^+(E) = \max \sigma\left(\mathcal{R}(E)\right)$. We claim

$$\lim_{E \downarrow B_n} r^+(E) = \infty. \tag{B.7}$$

To see this, let $\Pi_n = \chi_{\{B_n\}}(H_B)$. Then

$$\mathcal{R}(E) = \tfrac{1}{E - B_n}\sqrt{V^{(\Lambda)}_\omega}\,\Pi_n\sqrt{V^{(\Lambda)}_\omega} - \sqrt{V^{(\Lambda)}_\omega}\,(1 - \Pi_n)\,(H_B - E)^{-1}\sqrt{V^{(\Lambda)}_\omega}. \tag{B.8}$$

Since

$$\left\| \sqrt{V_\omega^{(\Lambda)}}\,(1-\Pi_n)\,(H_B - E)^{-1}\sqrt{V_\omega^{(\Lambda)}} \right\| \le \frac{M_2}{B} \quad \text{for} \quad E \in (B_n, B_n + B), \tag{B.9}$$

(B.7) follows if we show that $\sqrt{V_\omega^{(\Lambda)}}\Pi_n\sqrt{V_\omega^{(\Lambda)}} \neq 0$. But otherwise we would conclude that $\sqrt{V_\omega^{(\Lambda)}}\Pi_n = 0$ ($A^*A = 0$ implies $A = 0$), and, since $V_\omega^{(\Lambda)} > 0$ in an nonempty open set, we would contradict the unique continuation principle. Now, using (B.7), we conclude, as in [FK4, Proposition 4.3], that $H_{B,\lambda,\omega}^{(\Lambda)}$ has an eigenvalue in $(B_n, B_n + \lambda M_2]$ for all sufficiently small $\lambda > 0$.

Now, let us replace ω by M_2 in the notation if $\omega_j = M_2$ for all j, and consider $H_{B,\lambda,M_2}^{(\Lambda)}$. Fix $n \in \mathbb{N}$, and let $E_+^{(\Lambda)}(n, B, \lambda)$ denote the biggest eigenvalue of $H_{B,\lambda,M_2}^{(\Lambda)}$ in the open interval (B_n, B_{n+1}). We have shown the existence of $E_+^{(\Lambda)}(n, B, \lambda)$ for small $\lambda > 0$. By the argument in [K, Section VII.3.2], $E_+^{(\Lambda)}(n, B, \lambda)$ then exists for $\lambda \in (0, \lambda_+^{(\Lambda)}(n, B))$, with $\lambda_+^{(\Lambda)}(n, B) > 0$, where it is continuous and increasing in λ. In view of (B.5), we have $\lim_{\lambda\downarrow 0} E_+^{(\Lambda)}(n, B, \lambda) = B_n$ and $\lambda_+^{(\Lambda)}(n, B) \ge \frac{2B}{M_2}$. In addition, we must either have $\lambda_+^{(\Lambda)}(n, B) = \infty$ or $\lim_{\lambda\uparrow\lambda_+^{(\Lambda)}(n,B)} E_+^{(\Lambda)}(n, B, \lambda) = B_{n+1}$. In the latter case we may thus extend $E_+^{(\Lambda)}(n, B, \lambda)$ as an increasing, continuous function for $\lambda \in (0, \infty)$ by setting $E_+^{(\Lambda)}(n, B, \lambda) = B_{n+1}$ for $\lambda \ge \lambda_+^{(\Lambda)}(n, B)$.

A similar argument produces a smallest eigenvalue $E_-^{(\Lambda)}(n, B, \lambda) \in [B_{n-1}, B_n)$ of $H_{B,\lambda,-M_1}^{(\Lambda)}$ in (B_{n-1}, B_n) for $\lambda \in (0, \lambda_-^{(\Lambda)}(n, B))$, where $\lambda_-^{(\Lambda)}(n, B) \ge \frac{2B}{M_1}$, continuous and decreasing in λ, with $\lim_{\lambda\downarrow 0} E_-^{(\Lambda)}(n, B, \lambda) = B_n$. Moreover, $\lambda_-^{(\Lambda)}(1, B) = \infty$, and, for $n = 2, 3, \ldots$, either $\lambda_-^{(\Lambda)}(n, B) = \infty$ or $\lim_{\lambda\uparrow\lambda_-^{(\Lambda)}(n,B)} E_-^{(\Lambda)}(n, B, \lambda) = B_{n-1}$. In the latter case we extend $E_-^{(\Lambda)}(n, B, \lambda)$ as a decreasing, continuous function for $\lambda \in (0, \infty)$ by setting $E_-^{(\Lambda)}(n, B, \lambda) = B_{n-1}$ for $\lambda \ge \lambda_-^{(\Lambda)}(n, B)$.

For an arbitrary $\omega \in \Omega_{\text{supp}}$ and $\lambda > 0$, the eigenvalues of $H_{B,\lambda,\omega}^{(\Lambda)}$ in the intervals $(B_n, B_n + \lambda M_2)$ and $(B_n - \lambda M_1, B_n)$ (if they exist) are separately continuous and increasing in each $\omega_j \in [-M_1, M_2]$, $j \in \Lambda$, and hence they must be in the interval $I_n^{(\Lambda)}(B, \lambda) = [E_-^{(\Lambda)}(n, B, \lambda), E_+^{(\Lambda)}(n, B, \lambda)]$. Thus we conclude that for each square Λ we have

$$\bigcup_{\omega\in\Omega_{\text{supp}}} \sigma\left(H_{B,\lambda,\omega}^{(\Lambda)}\right) = \bigcup_{n\in\mathbb{N}} I_n^{(\Lambda)}(B, \lambda). \tag{B.10}$$

In addition, the same argument shows that for fixed λ and B we have $\pm E_\pm^{(\Lambda)}(n, B, \lambda)$ increasing with Λ. We set $E_+(n, B, \lambda) := \sup_\Lambda E_+^{(\Lambda)}(n, B, \lambda) \le B_{n+1}$, $E_-(n, B, \lambda) := \inf_\Lambda E_-^{(\Lambda)}(n, B, \lambda) \ge B_{n-1}$, and conclude from (B.4) and (B.10) that (cf. [GKS, Eq. (2.11)]

$$\Sigma_{B,\lambda} = \bigcup_{n\in\mathbb{N}} I_n(B, \lambda), \quad \text{where} \quad I_n(B, \lambda) = [E_-(n, B, \lambda), E_+(n, B, \lambda)]. \tag{B.11}$$

Note that the intervals $I_n(B, \lambda)$ depend on $\operatorname{supp}\mu = [-M_1, M_2]$, but not on other details of the measure μ.

Now assume that u in (2.6) satisfies

$$0 < U_- \leq U(x) := \sum_{i \in \mathbb{Z}^2} u(x-i) \leq 1, \tag{B.12}$$

for some constant U_-. (The upper bound is simply a normalization we had already assumed.) In this case, for all $n \in \mathbb{N}$ we have

$$B_n + \lambda M_2 U_- \leq E_+(n, B, \lambda) \quad \text{for} \quad \lambda \in \left(0, \tfrac{2B}{M_2 U_-}\right), \tag{B.13}$$

$$B_n - \lambda M_1 U_- \geq E_-(n, B, \lambda) \quad \text{for} \quad \lambda \in \left(0, \tfrac{2B}{M_1 U_-}\right). \tag{B.14}$$

We also have

$$B - \lambda M_1 U_- \geq E_-(1, B, \lambda) \quad \text{for all} \quad \lambda \geq 0. \tag{B.15}$$

This can be seen as follows. Take $\lambda \in (0, \frac{2B}{M_2 U_-})$, then

$$H_{B,\lambda,M_2} = H_B + \lambda M_2 U_- + \lambda M_2 (U - U_-), \text{ with } 0 \leq U - U_- \leq 1 - U_-. \tag{B.16}$$

Since $\sigma\left(H_B + \lambda M_2 U_-\right) = \Sigma_B + \lambda M_2 U_- = \{B_n + \lambda M_2 U_-;\ n \in \mathbb{N}\}$, it follows from [K, Theorem 4.10] (as in Lemma A.1), and the definition of $E_+(n, B, \lambda)$, that

$$\sigma\left(H_{B,\lambda,M_2}\right) \subset \bigcup_{n=1}^{\infty} [B_n + \lambda M_2 U_-, E_+(n, B, \lambda)]. \tag{B.17}$$

Since by the same argument

$$\Sigma_B + \lambda M_2 U_- \subset \bigcup_{n \in \mathbb{N}_{\neq \emptyset}} [B_n + \lambda M_2 U_- - \lambda M_2 (1 - U_-), E_+(n, B, \lambda)], \tag{B.18}$$

where $\mathbb{N}_{\neq\emptyset} := \{n \in \mathbb{N};\ \sigma\left(H_{B,\lambda,M_2}\right) \cap [B_n + \lambda M_2 U_-, E_+(n, B, \lambda)] \neq \emptyset\}$, we conclude that $\mathbb{N}_{\neq\emptyset} = \mathbb{N}$. It then follows from (B.11) that (B.13) holds. (B.14) and (B.15) are proved in a similar manner.

Under the condition (2.31) the spectral gaps never close. On the other hand, if we have (B.12), if

$$\lambda U_- (M_1 + M_2) \geq 2B, \tag{B.19}$$

all the internal spectral gaps close, i.e.,

$$\Sigma_{B,\lambda} = \left(E_-(1, B, \lambda), \infty\right). \tag{B.20}$$

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